Net Change and Displacement

We have seen that the definite integral $\int_a^b f(x)\,dx$ measures the *net area* under the curve $y = f(x)$ on the interval $[a,b]$. Any part of the region below the x-axis contributes negatively-signed area to this net calculation. To find the *total area* enclosed by f on $[a,b]$, one needs to evaluate the definite integral of the absolute value of $f(x)$: $\int_a^b |f(x)|\,dx$

We can apply this idea to other contexts. In Calculus I we interpreted the first and second derivatives as velocity and acceleration in the context of motion. In particular, l

- $s(t) =$ position at time t.
- $s'(t) = v(t) =$ velocity at time t.
- $s''(t) = v'(t) = a(t) =$ acceleration at time t.

Reversing our point of view and using antiderivatives, since position $s(t)$ is an antiderivative of veloctty $v(t)$, by the FTC (II) we have

$$\int_a^b v(t)\,dt = s(t)\Big|_a^b = s(b) - s(a).$$

But $s(b) - s(a)$ represents the *net distance traveled* or *displacement* of the object during the time interval because it is the difference in locations at the end and start of the time period.

If we wanted the *total distance traveled* rather than the net distance traveled, just as with the area problem, we would compute the integral of $|v(t)|$ instead. That is,

$$\text{total distance traveled} = \int_a^b |v(t)|\,dt.$$

This can be summarized as follows.

DEFINITION 6.1.1 (Displacement versus Distance). Assume that the **position** of an object moving along a straight line at time t is denoted by $s(t)$ relative to the origing and that its **velocity** is denoted by $v(t)$. Then

1. The **displacement** or net change in position of the object between times $t = a$ and $t = b > a$ is given by
$$s(b) - s(a) = \int_a^b v(t)\,dt.$$
2. The **total distance traveled** by the object between times $t = a$ and $t = b > a$ is given by
$$\int_a^b |v(t)|\,dt.$$

EXAMPLE 6.1.2. Suppose an object moves with velocity $v(t) = 2t^2 - 12t + 16$ km/hr along a straight road.

(1) Determine the displacement of the object on the time interval $[1,3]$ and $[0,4]$ and interpret your answer.

(2) Determine the distance travelled on the time interval $[0,4]$.

Solution.

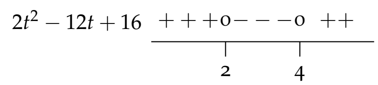
(1) The displacement is easy to calculate: For the interval $[0,4]$, On $[0,4]$,

$$\text{Displacement on } [0,4] = s(4) - s(0) = \int_0^4 2t^2 - 12t + 16\,dt$$
$$= \frac{2t^3}{3} - 6t^2 + 16t\Big|_0^4 = \frac{128}{3} - 96 + 64 - 0 = \frac{32}{3}.$$

On $[1,3]$,

$$s(3) - s(1) = \int_1^3 2t^2 - 12t + 16\,dt = \frac{2t^3}{3} - 6t^2 + 16t\Big|_1^4 = (18 - 54 + 48) - \left(\frac{2}{3} - 6 + 16\right) = \frac{4}{3}.$$

(2) The distance travelled is harder to determine since we need to integrate $|v(t)|$. We must first determine where $v(t)$ is positive and negative.



$$2t^2 - 12t + 16 = 2(t^2 - 6t + 8) = 2(t-2)(t-4) = 0 \Rightarrow t = 2 \text{ or } t = 4.$$

The number line to the right shows that $2t^2 - 12t + 16 \leq 0$ only $[2,4]$. We can now find the distance travelled (total area) by splitting the interval into two pieces $[0,2]$ and $[2,4]$, changing the sign of $v(t)$ on the second piece to obtain the absolute value of $v(t)$.

$$\text{Dist Trav} = \text{Total Area} = \int_0^4 |2t^2 - 12t + 16|\,dt$$
$$= \int_0^2 2t^2 - 12t + 16\,dt - \int_2^4 2t^2 - 12t + 16\,dt$$
$$= \left[\frac{2t^3}{3} - 6t^2 + 16t\right]\Big|_0^2 - \left[\frac{2t^3}{3} - 6t^2 + 16t\right]\Big|_2^4$$
$$= \frac{40}{3} + \frac{8}{3} = 16.$$

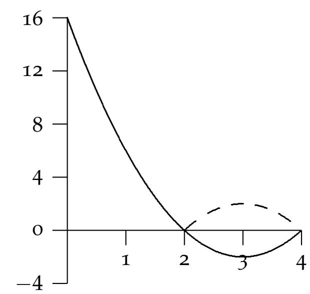


Figure 6.1: The distance travelled on $[0,4]$ is the area under the absolute value of the velocity curve.

EXAMPLE 6.1.3. Suppose an object moves with velocity $t^3 - 5t^2 + 4t$ m/s.

(1) Determine the displacement of the object on the time interval $[0,6]$ and interpret your answer.

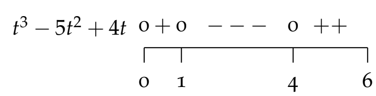
(2) Determine the distance travelled on $[0,6]$

Solution.

(1) For displacement on $[0,6]$,

$$s(6) - s(0) = \int_0^6 t^3 - 5t^2 + 4t\,dt = \frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\Big|_0^6 = (324 - 360 + 72) - 0 = 36.$$

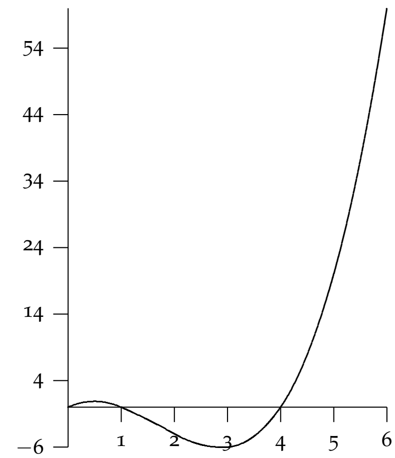
(2) For the distance travelled we must first determine where $v(t)$ is positive and negative.



$$t^3 - 5t^2 + 4t = t(t^2 - 5t + 4) = t(t-1)(t-4) = 0 \Rightarrow t = 0, 1, 4.$$

The number line to the right shows that $t^3 - 5t^2 + 4t \leq 0$ only $[1,4]$. We can now find the distance travelled (total area) by splitting the interval into three pieces $[0,1]$, $[1,4]$ and $[4,6]$, changing the sign of $v(t)$ on the second piece to obtain the absolute value of $v(t)$.

$$\begin{aligned}\text{Dist Trav} &= \int_0^6 |2t^2 - 12t + 16|\,dt \\ &= \int_0^1 t^3 - 5t^2 + 4t\,dt - \int_1^4 t^3 - 5t^2 + 4t\,dt + \int_4^6 t^3 - 5t^2 + 4t\,dt \\ &= \left[\frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\right]\Big|_0^1 - \left[\frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\right]\Big|_1^4 + \left[\frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\right]\Big|_4^6 \\ &= \frac{7}{12} + \frac{45}{4} + \frac{140}{3} = \frac{117}{2}.\end{aligned}$$



Future Positions

Suppose that we know the velocity of an object moving along a straight line is $v(x)$ and we know its position $s(a)$ at some time $t = a$. [Note: Often we know the *initial position* $s(0)$.] We can determine the future position at a time general time t using the displacement equation. Since $s(x)$ is an antiderivative of $v(x)$, again FTC (II) tells us that

$$\int_a^t v(x)\,dx = s(x)\Big|_a^t = s(t) - s(a).$$

Solving for the position $s(t)$, we see

THEOREM 6.1.4 (Position from Velocity)**.**

$$\text{Position at time } t = s(t) = s(a) + \int_a^t v(x)\,dx.$$

In a similar way, if $a(t)$ represents acceleration of the object,

$$\text{velocity at time } t = v(t) = v(a) + \int_a^t a(x)\,dx.$$

EXAMPLE 6.1.5. Suppose that the acceleration of an object is given by $a(x) = 2 - \cos x$ for $x \geq 0$ with

- $v(0) = 1$
- $s(0) = 3$.

Find $s(t)$.

Solution. First find $v(t)$ using Theorem 6.1.4.

$$v(t) = v(0) + \int_0^t a(x)\,dx = 1 + \int_0^t 2 - \cos x\,dt = 1 + [2x - \sin x]\Big|_0^t = 1 + (2t - \sin t) - (0) = 1 + 2t - \sin t.$$

Now solve for $s(t)$ by using Theorem 6.1.4.

$$\begin{aligned}s(t) = s(0) + \int_0^t v(x)\,dx = 3 + \int_0^t 1 + 2x - \sin x\,dx &= 3 + (x + x^2 + \cos x)\Big|_0^t \\ &= 3 + (t + t^2 + \cos t) - (0 + 0 + 1) = 2 + 2t + t^2 + \cos t.\end{aligned}$$

EXAMPLE 6.1.6. If acceleration is given by $a(t) = 10 + 3t - 3t^2$, find the exact position function, if $s(0) = 1$ and $s(2) = 11$.

Solution. This time we don't have a velocity at time 0. So let $v(0) = v_0$ be some unknown constant. We will see if we can work it out later. Then

$$v(t) = v(0) = \int_0^t a(x)\,dx = v_0 + \int_0^t 10 + 3x - 3x^2\,dx = v_0 + 10x + \tfrac{3}{2}x^2 - x^3\Big|_0^t = v_0 + 10t + \tfrac{3}{2}t^2 - t^3.$$

Now

$$s(t) = s(0) + \int_0^t v(x)\,dx = 1 + \int_0^t v_0 + 10x + \tfrac{3}{2}x^2 - x^3 + c\,dx$$
$$= 1 + (v_0x + 5x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4)\Big|_0^t = 1 + v_0t + 5t^2 + \tfrac{1}{2}t^3 - \tfrac{1}{4}t^4.$$

So $s(t) = 1 + v_0t + 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4$. To solve for v_0, evaluate at $s(t)$ at $t = 2$.

$$s(2) = 1 + 2v_0 + 20 + 4 - 4 = 11 \text{ so } 2v_0 = -10 \Rightarrow v_0 = -5.$$

Thus, $s(t) = 1 - 5t + 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4$.

Constant Acceleration: Gravity

In many motion problems the acceleration is constant. This happens when an object is thrown or dropped and the only acceleration is due to gravity. In such a situation we have

- $a(t) = a$, constant acceleration
- with initial velocity $v(0) = v_0$
- and initial position $s(0) = s_0$.

Then using Theorem 6.1.4

$$v(t) = v(0) + \int_0^t a(x)\,dx = v_0 + \int_0^t a\,ds = v_0 + ax\Big|_0^t = v_0 + (at - 0) = at + v_0.$$

So

$$\boxed{v(t) = at + v_0.}$$

Next,

$$s(t) = s(0) + \int_0^t v(x)\,dx = s_0 + \int_0^t as + v_0\,ds = s_0 + (\tfrac{1}{2}ax^2 + v_0x)\Big|_0^t = s_0 + (\tfrac{1}{2}at^2 + v_0t - 0) = \tfrac{1}{2}at^2 + v_0t + s_0.$$

Therefore

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0t + s_0.}$$

EXAMPLE 6.1.7. Suppose a ball is thrown with initial velocity 96 ft/s from a roof top 432 feet high. The acceleration due to gravity is constant $a(t) = -32$ ft/s^2. Find $v(t)$ and $s(t)$. Then find the maximum height of the ball and the time when the ball hits the ground.

Solution. Recognizing that $v_0 = 96$ and $s_0 = 432$ and that the acceleration is constant, we may use the general formulas we just developed.

$$\boxed{v(t) = at + v_0 = -32t + 96}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = -16t^2 + 96t + 432.}$$

The max height occurs when the velocity is 0 (when the ball stops rising):

$$v(t) = -32t + 96 = 0 \Rightarrow t = 3 \Rightarrow s(3) = -144 + 288 + 432 = 576 \text{ ft}.$$

The ball hits the ground when $s(t) = 0$.

$$s(t) = -16t^2 + 96t + 432 = -16(t^2 - 6t - 27) = -16(t-9)(t+3) = 0.$$

So $t = 9$ only (since $t = -3$ does not make sense).

EXAMPLE 6.1.8. A person drops a stone from a bridge. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

Solution. Here's what we know. $v_0 = 0$ (dropped) and $s(5) = 0$ (hits water). And we know acceleration is constant, $a = -32$ ft/s^2. We want to find the height of the bridge, which is just s_0. Use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = -32t}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + s_0.}$$

Now we use the position we know: $s(5) = 0$.

$$s(5) = -16(5)^2 + s_0 \Rightarrow s_0 = 400 \text{ ft}.$$

Notice that we did not need to use the velocity function.

YOU TRY IT 6.1 (Extra Credit). In the previous problem we did not take into account that sound does not travel instantaneously in your calculation above. Assume that sound travels at 1120 ft/s. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

Check on your answer: Should the bridge be higher or lower than in the preceding example? Why?

EXAMPLE 6.1.9. Here's a variation. This time we will use metric units. Suppose a ball is thrown with unknown initial velocity v_0 m/s from a roof top 49 meters high and the position of the ball at time $t = 3$ is $s(3) = 0$. The acceleration due to gravity is constant $a(t) = -9.8$ m/s^2. Find $v(t)$ and $s(t)$.

Solution. This time v_0 is unknown but $s_0 = 49$ and $s(3) = 0$. Again the acceleration is constant so we may use the general formulas for this situation.

$$v(t) = at + v_0 = -9.8t + v_0$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -4.9t^2 + v_0 t + 49.$$

But we know that

$$s(3) = -4.9(3)^2 + v_0 \cdot 3 + 49 = 0$$

which means

$$3v_0 = 4.9(9) - 4.9(10) = -4.9 \Rightarrow v_0 = -4.9/3.$$

So

$$\boxed{v(t) = -9.8t - \tfrac{49}{30}}$$

and

$$\boxed{s(t) = -4.9t^2 - \tfrac{49}{30}t + 49.}$$

Interpret $v_0 = -4.9/3$.

More on Net Change and Future Values

We've interpreted net area as displacement and total area as total distance when working with a velocity function. But we can apply these same ideas *anytime* we have a rate of change function. Examples include when $f(t)$ is a flow rate of liquid (in which case we can compute net change in volume), or $f(t)$ is a population growth rate (in which case we can compute future population estimates), or $p(t)$ is the growth of a financial account (in which case we might compute the net change in value of the account).

In general, suppose we know the rate of change in some quantity is given by $Q'(x)$ on the interval $[0, t]$. Then by integrating using FTC II and the fact that $Q(x)$ is an antiderivative of $Q'(x)$, we get

$$\int_0^t Q'(x)\,dx = Q(x)\Big|_0^t = Q(t) - Q(0).$$

Again, as in Theorem 6.1.4, we can rearrange terms to get

THEOREM 6.1.10 (Net Change and Future Value). Assume that a quantity Q changes over time at a know rate Q'. Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$Q(b) - Q(a) = \int_a^b Q'(t)\,dt.$$

Further, if $Q(0)$ is the initial value, then the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx.$$

EXAMPLE 6.1.11 (Population Growth). Suppose a rafter of wild turkeys in Geneva has an initial value of $P(0) = 23$ and the community grows at a rate of $P'(t) = 20 - 3\sqrt{t}$, where time is measured in years. Determine the population in 4 years. Then find the general formula for the population, $P(t)$.

A group of turkeys is called a rafter. A group of turtledoves is called a pitying.

Solution. By the second part of Theorem 6.1.10,

$$P(4) = P(0) + \int_0^4 P'(t)\,dt = 23 + \int_0^6 15 - \sqrt{t}\,dt$$
$$= 23 + \left(20t - \frac{6t^{3/2}}{3}\right)\Bigg|_0^4 = 23 + (80 - 16) - (0) = 87.$$

More generally, the population at time t is

$$P(t) = P(0) + \int_0^t P'(x)\,dx = 23 + \int_0^t 15 - \sqrt{x}\,dx$$
$$= 23 + \left(20x - \frac{6x^{3/2}}{3}\right)\Bigg|_0^t = 23 + 20t - \frac{6t^{3/2}}{3}.$$

EXAMPLE 6.1.12 (Savings). Grandparents of a newborn deposit \$10,000 in a college savings account that has a growth rate of $555e^{0.055t}$. How much will be available in the account when the child is 18 and starting college?

Solution. By the second part of Theorem 6.1.10,

$$Q(18) = Q(0) + \int_0^{18} Q'(t)\,dt = 10,000 + \int_0^{18} 555e^{0.0555t}\,dt$$
$$= 10,000 + \frac{555}{0.0555}e^{0.0555t}\Big|_0^{18}$$
$$= 10,000 + 10,000e^{0.0555t}\Big|_0^{18}$$
$$= 10,000 + (10,000e^{0.999} - 10,000) = 27,155.65$$

Not bad, but not worth a semester at HWS.

EXAMPLE 6.1.13 (Economics). The **marginal cost** of a product is additional the cost incurred by producing one more item of the product. Typically, as production increases, the marginal cost decreases, at least up to a point. This is what people mean by the term "economy of scale." Marginal cost is approximated by the derivative of the cost function $C'(x)$ that depends on the number of units x being produced.

Suppose that the marginal cost is given by the function $C'(x) = 200 + 10x - 0.01x^2$. Find the additional cost incurred when production rises from 500 to 550 units. Then find the cost incurred when production rises from 550 to 600 units.

Solution. By the first part of Theorem 6.1.10,

$$\begin{aligned} C(550) - C(500) = \int_{500}^{550} C'(x)\,dx &= \int_{500}^{550} 300 + 10x - 0.01x^2\,dx \\ &= 300x + 5x^2 - \frac{0.01x^3}{3}\Big|_{500}^{550} \\ &= \left(300(550) + 5(550)^2 - \frac{0.01(550)^3}{3}\right) - \left(300(500) + 5(500)^2 - \frac{0.01(500)^3}{3}\right) \\ &= 139,583. \end{aligned}$$

For the second part

$$\begin{aligned} C(600) - C(550) = \int_{550}^{600} C'(x)\,dx &= \int_{550}^{600} 300 + 10x - 0.01x^2\,dx \\ &= 300x + 5x^2 - \frac{0.01x^3}{3}\Big|_{550}^{600} \\ &= \left(300(600) + 5(600)^2 - \frac{0.01(600)^3}{3}\right) - \left(300(550) + 5(550)^2 - \frac{0.01(550)^3}{3}\right) \\ &= 137,083. \end{aligned}$$

Notice that the cost making 50 more units has decreased, illustrating the idea that marginal cost decreases as production increases.

For Fun: Additional Motion Problems with Constant Acceleration

The following problems all involve motion with constant acceleration. When acceleration is constant, we can use the equations we developed a few pages earlier. But remember, when acceleration is not constant, be sure to use Theorem 6.1.4.

EXAMPLE 6.1.14. Mo Green is attempting to run the 100m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his *acceleration* is constant, a, over the entire race. Determine his velocity function. (a will still appear as an unknown constant.) Determine his position function. There should be no unknown constants in your equation at this point. What is his velocity at the end of the race? Do you think this is realistic?

Solution. We have: constant acceleration $= a\,\text{m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2} = 2.0825\,\text{m/s}^2$. So $s(t) = 2.0825t^2$. Mo's velocity at the end of the race is $v(9.8) = a \cdot 9.8 = 2.0825(9.8) = 20.41$ m/s...not realistic.

EXAMPLE 6.1.15. A stone dropped off a cliff hits the ground with speed of 120 ft/s. What was the height of the cliff?

Solution. Notice that $v_0 = 0$ (dropped!) and s_0 is unknown but is equal to the cliff height, and that the acceleration is constant $a = -32$ ft/. Use the general formulas for motion with constant acceleration:

$$\boxed{v(t) = at + v_0 = -32t + 0 = -32t}.$$

Now we use the velocity function and the one velocity value we know: $v = -120$ when it hits the ground. So the *time* when it hits the ground is given by

$$v(t) = -32t = -120 \Rightarrow t = 120/32 = 15/4$$

when it hits the ground. Now remember when it hits the ground the height is 0. So $s(15/4) = 0$. But we know

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + 0t + s_0 = -16t^2 + s_0.}$$

Now substitute in $t = 15/4$ and solve for s_0.

$$s(15/4) = 0 \Rightarrow -16(15/4)^2 + s_0 = 0 \Rightarrow s_0 = 15^2 = 225.$$

The cliff height is 225 feet.

EXAMPLE 6.1.16. A car is traveling at 90 km/h when the driver sees a deer 75 m ahead and slams on the brakes. What constant deceleration is required to avoid hitting Bambi? [Note: First convert 90 km/h to m/s.]

Solution. Let's list all that we know. $v_0 = 90$ km/h or $\frac{90000}{60 \cdot 60} = 25$ m/s and $s_0 = 0$. Let time t^* represent the time it takes to stop. Then $s(t^*) = 75$ m. Now the car is stopped at time t^*, so we know $v(t^*) = 0$. Finally we know that acceleration is an unknown constant, a, which is what we want to find.

Now we use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = at + 25}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2 + 25t.}$$

Now use the other velocity and position we know: $v(t^*) = 0$ and $s(t^*) = 75$ when the car stops. So

$$v(t^*) = at^* + 25 = 0 \Rightarrow t^* = -25/a$$

and

$$s(t^*) = \tfrac{1}{2}a(t^*)^2 + 25t^* = \tfrac{1}{2}a(-25/a)^2 + 25(-25/a) = 75.$$

Simplify to get

$$\frac{625a}{2a^2} - \frac{625}{a} = \frac{625}{2a} - \frac{1350}{2a} = -\frac{625}{2a} = 75 \Rightarrow 150a = -625$$

so

$$a = -\frac{625}{150} = -\frac{25}{6} \text{ m/s}.$$

(Why is acceleration negative?)

EXAMPLE 6.1.17. One car intends to pass another on a back road. What constant acceleration is required to increase the speed of a car from 30 mph (44 ft/s) to 50 mph ($\frac{220}{3}$ ft/s) in 5 seconds?

Solution. Given: $a(t) = a$ constant. $v_0 = 44$ ft/s. $s_0 = 0$. And $v(5) = \frac{220}{3}$ ft/s. Find a. But

$$v(t) = at + v_0 = at + 44.$$

So

$$v(5) = 5a + 44 = \frac{220}{3} \Rightarrow 5a = \frac{220}{3} - 44 = \frac{88}{3}.$$

Thus $a = \frac{88}{15}$.

YOU TRY IT 6.2. A toy bumper car is moving back and forth along a straight track. Its acceleration is $a(t) = \cos t + \sin t$. Find the particular velocity and position functions given that $v(\pi/4) = 0$ and $s(\pi) = 1$.

ANSWER TO YOU TRY IT 6.2. $v(t) = \int a(t)\,dt = \int \cos t + \sin t\,dt = \sin t - \cos t + c$. So $v(\pi/4) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + c = 0 \Rightarrow c = 0$. Thus, $v(t) = \sin t - \cos t$. Now $s(t) = \int v(t)\,dt = \int \sin t - \cos t\,dt = -\cos t - \sin t + c$. Since $s(\pi) = -(-1) - 0 + c = 1 \Rightarrow c = 0$. So $s(t) = -\cos t - \sin t$.

WEBWORK: Click to try Problems 71 through 74. Use GUEST login, if not in my course.