7.3 A Complication: Higher Degrees

The first complication that arises is is that the denominator of the rational function may factor into three or more distinct linear factors. The solution method works the same way as above, but it may be more complicated to find the constants.

EXAMPLE 7.3.1 (Partial Fractions: Three Distinct Linear Factors). Determine

$$\int \frac{2x^2 - 6x + 2}{x^3 - 3x^2 + 2x}\, dx.$$

Solution. Check that this is not quite a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into distinct three linear factors: $x(x^2 - 3x + 2) = x(x-1)(x-2)$. We form the partial fractions with a different constant for each linear factor:

$$\begin{aligned}\frac{2x^2 - 6x + 2}{x^3 - 3x^2 + 2x} = \frac{2x^2 - 6x + 2}{x(x-1)(x-2)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \\ &= \frac{A(x-1)(x-2) + Bx(x-2) + Cx(x-1)}{x^3 - 3x^2 + 2x} \\ &= \frac{Ax^2 - 3Ax + 2A + Bx^2 - 2B + Cx^2 - Cx}{x^3 - 3x^2 + 2x}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{lrcll rcr} x^2\text{'s:} & 2 & = & A + B + C & & \Rightarrow\ 1 & = & B + C \\ x\text{'s:} & -6 & = & -3A - 2B - C & & \Rightarrow\ -3 & = & -2B - C \\ \hline \text{const:} & 2 & = & 2A & \Rightarrow A = 1 & -2 & = & -B \end{array}$$

Notice that we used the value $A = 1$ to simplify the first two equations. It follows that $B = 2$ and $C = -1$. The integration becomes:

$$\int \frac{2x^2 - 6x + 2}{x^3 - 3x^2 + 2x}\, dx = \int \frac{1}{x} + \frac{2}{x-1} - \frac{1}{x-2}\, dx = \ln|x| + 2\ln|x-1| - \ln|x-2| + c.$$

YOU TRY IT 7.3. What would the numerator in the original integral have to be to make the problem a substitution problem?

EXAMPLE 7.3.2 (Partial Fractions: Three Distinct Linear Factors). Determine

$$\int \frac{x^2 + 4x - 1}{x^3 - x}\, dx.$$

Solution. Check that this is not a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into distinct three linear factors: $x(x^2 - 1) = x(x-1)(x+1)$. We form the partial fractions with a different constant for each linear factor:

$$\begin{aligned}\frac{x^2 + 4x - 1}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} &= \frac{A(x^2 - 1) + Bx(x+1) + Cx(x-1)}{x^3 - x} \\ &= \frac{Ax^2 - A + Bx^2 + B + Cx^2 - Cx}{x^3 - x}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{lrcll rcr} x^2\text{'s:} & 1 & = & A + B + C & & \Rightarrow\ 0 & = & B + C \\ x\text{'s:} & 4 & = & B - C & & \Rightarrow\ 4 & = & B - C \\ \hline \text{const:} & -1 & = & -A & \Rightarrow A = 1 & 4 & = & 2B \end{array}$$

Notice that we used the value $A = 1$ to simplify the first two equations. It follows that $B = 2$ and $C = -2$. The integration becomes:

$$\int \frac{x^2+4x-1}{x^3-x}\,dx = \int \frac{1}{x} + \frac{2}{x-1} - \frac{2}{x+1}\,dx = \ln|x| + 2\ln|x-1| - 2\ln|x+1| + c$$
$$= \ln|x| + 2\ln\left|\frac{x-1}{x+1}\right| + c.$$

EXAMPLE 7.3.3 (Partial Fractions: Three Distinct Linear Factors). Determine

$$\int \frac{4x+28}{(x+1)(x^2-4x+3)}\,dx.$$

Solution. Check that this is not a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($1 < 3$) and the denominator factors into distinct three linear factors: $(x+1)(x^2-4x+3) = (x+1)(x-1)(x-3)$. We form the partial fractions with a different constant for each linear factor:

$$\frac{4x+28}{(x+1)(x^2-4x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-3} = \frac{A(x^2-4x+3)+B(x^2-2x-3)+C(x^2-1)}{(x+1)(x^2-4x+3)}$$
$$= \frac{(A+B+C)x^2+(-4A-2B)x+3A-3B-C}{(x+1)(x^2-4x+3)}.$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{llcll}
x^2\text{'s:} & 0 & = & A+B+C & \Rightarrow\ C=5 \\
x\text{'s:} & 4 & = & -4A-2B & \Rightarrow\ A=3 \\
\text{const:} & 28 & = & 3A-3B-C & \\
\hline
\text{Add all:} & 32 & = & -4B & \Rightarrow B=-8
\end{array}$$

The integration becomes:

$$\int \frac{4x+28}{(x+1)(x^2-4x+3)}\,dx = \int \frac{3}{x+1} - \frac{8}{x-1} + \frac{5}{x-3}\,dx$$
$$= 3\ln|x-1| - 8\ln|x+1| + 5\ln|x-3| + c$$

YOU TRY IT 7.4. What would the numerator in the original integral have to be to make the problem a substitution problem?

WEBWORK: Click to try Problems 119 through 120. Use GUEST login, if not in my course.

7.4 A Second Complication: Repeated Factors

Another complication that can arise is that the denominator of the rational function factors into linear factors, but the factors are not distinct. That is, there are repeated linear factors. The solution in this situation is to *include a term for every power of every linear factor that divides the denominator*.

EXAMPLE 7.4.1 (Partial Fractions: Repeated Linear Factors). Determine

$$\int \frac{3x^2-7x+2}{x^3-2x^2+x}\,dx.$$

Solution. This is not a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into repeated linear factors:

$$x(x^2 - 2x + 1) = x(x-1)(x-1) = x(x-1)^2.$$

We form the partial fractions with a different constant for each power of each factor that divides the denominator. Note: There are terms for both $x - 1$ and $(x-1)^2$. Warning! Look very carefully at the numerator in the step where the common denominator is formed.

$$\begin{aligned}\frac{3x^2 - 7x + 2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} &= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} \\ &= \frac{Ax^2 - 2Ax + A + Bx^2 - Bx + Cx}{x(x-1)^2}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{llll} x^2\text{'s:} & 3 & = & A + B & \qquad\qquad\qquad \Rightarrow 1 = B \\ x\text{'s:} & -7 & = & -2A - B + C & \qquad\qquad\qquad\qquad \Rightarrow C = -2 \\ \text{const:} & 2 & = & A & \qquad \Rightarrow A = 2 \end{array}$$

Notice that we used the value $A = 2$ to simplify the first two equations. The integration becomes:

$$\begin{aligned}\int \frac{3x^2 - 7x + 2}{x^3 - 2x^2 + x}\,dx &= \int \frac{2}{x} + \frac{1}{x-1} - \frac{2}{(x-1)^2}\,dx \\ &= 2\ln|x| + \ln|x-1| + 2(x-1)^{-1} + c.\end{aligned}$$

Be careful! The final integral requires the power rule and a mini-substitution.

YOU TRY IT 7.5. What would the numerator in the original integral have to be to make the problem a substitution problem?

EXAMPLE 7.4.2 (Partial Fractions: Repeated Linear Factors)**.** Determine

$$\int \frac{3x^2 - 2x - 3}{x^3 - x^2}\,dx.$$

Solution. This is not quite a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into repeated linear factors: $x^2(x-1)$. We form the partial fractions with a different constant for each power of each factor that divides the denominator—there are terms for both x and x^2. Look very carefully at the numerator in the step where the common denominator is formed.

$$\begin{aligned}\frac{3x^2 - 2x - 3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} &= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)} \\ &= \frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{llll} x^2\text{'s:} & 3 & = & A + C & \qquad\qquad\qquad \Rightarrow C = -2 \\ x\text{'s:} & -2 & = & -A + B & \qquad\qquad \Rightarrow A = 5 \\ \text{const:} & -3 & = & -B & \qquad \Rightarrow B = 3 \end{array}$$

Notice that we used the value $B = 3$ to simplify the first two equations. The integration becomes:

$$\int \frac{3x^2 - 2x - 3}{x^3 - x^2}\,dx = \int \frac{5}{x} + \frac{3}{x^2} - \frac{2}{x-1}\,dx = 5\ln|x| - 3x^{-1} - 2\ln|x-1| + c.$$

YOU TRY IT 7.6. What would the numerator in the original integral have to be to make the problem a substitution problem?

YOU TRY IT 7.7. Determine $\displaystyle\int \frac{7-x}{(x+1)(x-1)^2}\,dx$.

ANSWER TO YOU TRY IT 7.7. $2\ln|x+1| - 2\ln|x-1| - \frac{3}{x-1}$.

EXAMPLE 7.4.3 (Partial Fractions: Repeated Linear Factors). Here's a different one: Determine

$$\int \frac{x^2}{(x+1)^3}\,dx.$$

Solution. The degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into repeated linear factors: $(x+1)^3 = (x+1)(x+1)(x+1)$. We form the partial fractions with a different constant for each power of each factor that divides the denominator: $(x+1)$, $(x+1)^2$, and $(x+1)^3$. Look very carefully at the numerator in the step where the common denominator is formed.

$$\begin{aligned}\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} &= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} \\ &= \frac{Ax^2 + 2Ax + A + Bx + B + C}{(x+1)^3}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{lrcl l}
x^2\text{'s:} & 1 & = & A & \Rightarrow A = 1 \\
x\text{'s:} & 0 & = & 2A + B & \Rightarrow B = -2 \\
\text{const:} & 0 & = & A + B + C & \Rightarrow C = 1
\end{array}$$

Notice that we used the value $A = 1$ to simplify the last two equations. The integration becomes:

$$\begin{aligned}\int \frac{x^2}{(x+1)^3}\,dx &= \int \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\,dx \\ &= \ln|x+1| + 2(x+1)^{-1} - \tfrac{1}{2}(x+1)^{-2} + c.\end{aligned}$$

Alternative Solution. We could have used a u-substitution to solve this problem. Let $u = x+1$. Then $du = dx$. Since $u = x+1$, then $x = u-1$ so $x^2 = (u-1)^2 = u^2 - 2u + 1$. Therefore

$$\begin{aligned}\int \frac{x^2}{(x+1)^3}\,dx = \int \frac{u^2 - 2u + 1}{u^3}\,du &= \int \frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\,du \\ &= \ln|u| - 2u^{-1} - \tfrac{1}{2}u^{-2} + c \\ &= \ln|x+1| + 2(x+1)^{-1} - \tfrac{1}{2}(x+1)^{-2} + c,\end{aligned}$$

just as above. Which method would you have used? Why?

WEBWORK: Click to try Problems 121 through 124. Use GUEST login, if not in my course.

7.5 Problems

1. Try integrating these rational functions (answers below). These are a bit harder than those in the text. Some have three factors. Others have repeated factors.

(*a*) $\displaystyle\int \frac{4t^2-3t-4}{t^3-t^2-2t}\,dt$ (*b*) $\displaystyle\int \frac{t+7}{(t+1)(t^2-4t+3)}\,dt$ (*c*) $\displaystyle\int \frac{x+6}{x^2-x-6}\,dx$

(*d*) $\displaystyle\int \frac{x}{(x-1)(x+1)(x+2)}\,dx$ (*e*) $\displaystyle\int \frac{2x^2}{(x-1)^2(x+1)}\,dx$ (*f*) $\displaystyle\int \frac{-4x+4}{(x-2)^2x}\,dx$

(*g*) $\displaystyle\int \frac{2x}{(x-2)(x^2-1)}\,dx$ (*h*) $\displaystyle\int \frac{2x-1}{(x+2)^2}\,dx$

2. Try these similar looking problems.

(*a*) $\displaystyle\int \frac{10}{25+x^2}\,dx$ (*b*) $\displaystyle\int \frac{10x}{25+x^2}\,dx$ (*c*) $\displaystyle\int \frac{10}{25-x^2}\,dx$

3. Have you finished all the ones above? Do these similar looking integrals.

(*a*) $\displaystyle\int \frac{4}{\sqrt{4+x^2}}\,dx$ (*b*) $\displaystyle\int \frac{4}{\sqrt{4-x^2}}\,dx$

(*c*) $\displaystyle\int \frac{4x}{(4+x^2)^{3/2}}\,dx$ (*d*) $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$

Answers to Practice Problems

1. (*a*) $\int \frac{2}{t}+\frac{1}{t-2}+\frac{1}{t+1}\,dt = 2\ln|t|+\ln|t-2|+\ln|t+1|+c$

(*b*) $\int \frac{3/4}{t+1}+\frac{5/4}{t-3}-\frac{2}{t-1}\,dt = \frac{3}{4}\ln|t+1|+\frac{5}{4}\ln|t-3|-2\ln|t-1|+c$

(*c*) $\int \frac{9/5}{x-3}-\frac{4/5}{x+2}\,dt = x+\frac{9}{5}\ln|x-3|-\frac{4}{5}\ln|x+2|+c$

(*d*) $\int \frac{1/6}{x-1}+\frac{1/2}{x+1}-\frac{2/3}{x+2}\,dx = \frac{1}{6}\ln|x-1|+\frac{1}{2}\ln|x+1|-\frac{2}{3}\ln|x+2|+c$

(*e*) $\int \frac{3/2}{x-1}+\frac{1}{(x-1)^2}+\frac{1/2}{x+1}\,dx = \frac{3}{2}\ln|x-1|-(x-1)^{-1}+\frac{1}{2}\ln|x+1|+c$

(*f*) $\int -\frac{1}{x-2}-\frac{2}{(x-2)^2}+\frac{1}{x}\,dx = -\ln|x-2|+2(x-2)^{-1}+\ln|x|+c$

(*g*) $\int \frac{2}{x+2}-\frac{5}{(x+2)^2}\,dx = 2\ln|x+2|+\frac{5}{x+2}+c.$

2. All "$+c$".

(*a*) $2\arctan(x/5)$ (*b*) $5\ln|25+x^2|$ (*c*) $\ln\left|\dfrac{x+5}{x-5}\right|$