Power Series

So far we have looked at series of *numbers* such as $\sum_{n=1}^{\infty} \frac{1}{n}$ or $\sum_{n=1}^{\infty} \frac{n!}{10^n}$. While these are interesting, as calculus students, we are primarily interested in functions. Among the simplest functions are *polynomials* such as

$$p_1(x) = 1 + x$$
$$p_2(x) = 1 + x + \frac{x^2}{2!}$$
$$p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

What would happen if we looked at a polynomial of infinite degree, say

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

or expressed as a series

$$p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Each number x can 'plugged in' the series and we can determine whether $p(x)$ exists (the series converges for this x) or does not exist (the series diverges). If we do this for each and every x, we can determine the domain of $p(x)$. We will soon see that this can be done very efficiently.

The ultimate goal is to represent familiar functions like $\sin x$ and e^x as 'infinite degree' polynomials so that the values of these functions can be estimated quickly. This is how your calculator works!

DEFINITION 15.1.1 (Power Series). A **power series** in the variable x is a infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + a_n x^n + \cdots.$$

More generally, if a is some constant, then

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series centered at** a.
CONVENTION: To simplify notation, in the context of power series we define $x^0 = 1$ and $(x-a)^0 = 1$ for any value of x including $x = 0$.

EXAMPLE 15.1.2. Here are three simple examples:

1. $\sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is a power series (centered at 0).

2. $\sum_{n=1}^{\infty} 2^n(x-1)^n = 1 + 2(x-1) + 4(x-1)^2 + 8(x-1)^3 + \cdots$ is a power series centered at 1.

3. $\sum_{n=1}^{\infty} (-1)^n(x+2)^n = 1 - (x+2) + (x+2)^2 - (x+2)^3 + \cdots$ is a power series centered at -2.

Radius and Interval of Convergence

How do we determine where a power series converges? Do we have to check at every x? Notice that if we think of a power series as a function,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n,$$

there is at least one point x where the series *must* converge. Which?

At the center a the series will converge because (remember our convention)

$$f(a) = \sum_{n=0}^{\infty} c_n(a-a)^n = \sum_{n=0}^{\infty} a_n(0)^n = c_0(1) + 0 + 0 + \cdots = c_0.$$

So the center a of the power series is always in the domain of f.

The main theorem below tells us that the domain of a power series can only have three basic forms.

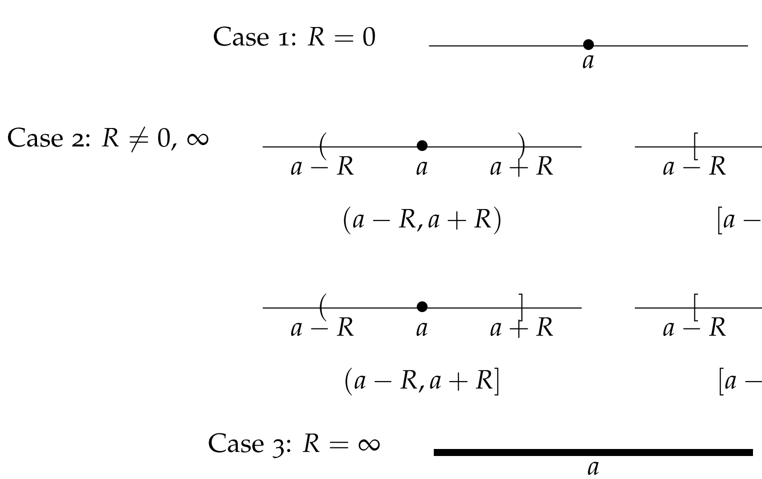
THEOREM 15.1.3 (Radius of Convergence). For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a, exactly one of the following is true:

1. The series converges only at a.
2. There is a number $R > 0$ so that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. (See NOTE below.)
3. The series converges for all x.

R is called the **radius of convergence**. If the series converges only at a, we say that $R = 0$. If the series converges at all values of x, we say that $R = \infty$.

The set of all values of x for which the series converges is called the **interval of convergence**.

NOTE: In case (2) the power series may converge at both endpoints, either endpoint, or neither endpoint. You have to check the convergence at the endpoints separately. Here's what the intervals of convergence can look like:

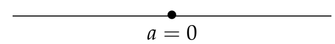


EXAMPLE 15.1.4. Determine the radius and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{n!x^n}{2^n}$.

Solution. We know that the series converges at its center $a = 0$. The standard methodology is to **use the ratio test** or the **root test** on the absolute values of the terms of the series. If we denote the nth term of the series by a_n, then for any $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!x^{n+1}}{2^{n+1}}}{\frac{n!x^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{2} \right| = |x| \lim_{n\to\infty} \frac{(n+1)}{2} = \infty.$$

By the ratio test, the series diverges for any $|x| > 0$. That is, it converges only at its center $a = 0$ and the radius of convergence is $R = 0$.



EXAMPLE 15.1.5. Determine the radius and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$.

Solution. We know that the series converges at its center $a = 2$. This time for any $x \neq 2$ (using reciprocals)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{x-2}{3} \right| = \frac{|x-2|}{3}.$$

By the ratio test, the series *converges* if

$$\frac{|x-2|}{3} < 1 \iff |x-2| < 3.$$

It also *diverges* when $|x-2| > 3$. The radius of convergence is $R = 3$. What about the endpoints $a - R = 2 - 3 = -1$ and $a + R = 2 + 3 = 5$?

For $x = -1$: This means we set $x = -1$ in the original series. We get

$$\sum_{n=0}^{\infty} \frac{(-1-2)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$$

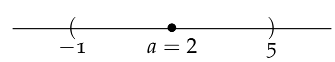
which diverges by the geometric series test ($|r| = 1$).

For $x = 5$: This means we set $x = 5$ in the original series. We get

$$\sum_{n=0}^{\infty} \frac{(5-2)^n}{3^n} = \sum_{n=0}^{\infty} (1)^n$$

which also diverges by the geometric series test ($|r| = 1$).

So the interval of convergence is $(-1, 5)$ and does not include either endpoint.



Interval of convergence: $(-1, 5)$

EXAMPLE 15.1.6. Determine the radius and interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$.

Solution. We know that the series converges at its center $a = 0$. This time for any $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot x \right| = |x|.$$

By the ratio test, the series *converges* if $|x| < 1$ and *diverges* when $|x| > 1$. The radius of convergence is $R = 1$. What about the endpoints $a - R = 0 - 1 = -1$ and $a + R = 0 + 1 = 1$?

For $x = -1$: We get

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

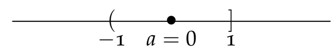
which diverges by the p-series test ($p = 1$).

For $x = 1$: We get

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is the alternating harmonic series which we have previously seen converges by the alternating series test. (This should be familiar. Try it!)

The interval of convergence is $(-1, 1]$ and includes just one of the endpoints.



Interval of convergence: $(-1, 1]$

EXAMPLE 15.1.7. Determine the radius and interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2}$.

Solution. We know that the series converges at its center $a = 0$. This time for any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n}\right| = \lim_{n\to\infty} \left|\frac{2n^2x}{(n+1)^2}\right| = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^2 \cdot 2x\right| = 2|x|.$$

By the ratio test, the series *converges* if $2|x| < 1 \iff |x| < \frac{1}{2}$ and *diverges* when $|x| > \frac{1}{2}$. The radius of convergence is $R = \frac{1}{2}$. What about the endpoints $-\frac{1}{2}$ and $\frac{1}{2}$?

For $x = -\frac{1}{2}$: We get

$$\sum_{n=0}^{\infty} \frac{(2(-\frac{1}{2}))^n}{n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

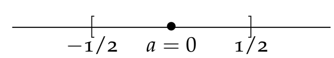
which converges by the alternating series test (an easy check!).

For $x = \frac{1}{2}$: We get

$$\sum_{n=0}^{\infty} \frac{(2(\frac{1}{2}))^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

which converges by the p-series test ($p = 2 > 1$).

The interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$ and includes both of the endpoints.



Interval of convergence: $[-\frac{1}{2}, \frac{1}{2}]$

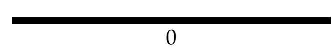
EXAMPLE 15.1.8. Determine the radius and interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. We know that the series converges at its center $a = 0$. This time for any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1.$$

By the ratio test, the series *converges* for all x. The radius of convergence is $R = \infty$. There are no endpoints to check.

The interval of convergence is $(-\infty, \infty)$.



Interval of convergence: $(-\infty, \infty)$

EXAMPLE 15.1.9. Determine the radius and interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3n^2+1}$.

Solution. We know that the series converges at its center $a = 0$. This time for any $x \neq 0$

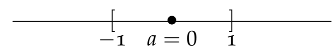
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{3(n+1)^2+1} \cdot \frac{3n^2+1}{x^n}\right| = \lim_{n\to\infty} \left|\frac{3n^2+1}{3n^2+6n+4} \cdot x\right| \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{3n^2}{3n^2} \cdot x\right| = |x|.$$

By the ratio test, the series *converges* if $|x| < 1$ and *diverges* when $|x| > 1$. The radius of convergence is $R = 1$. What about the endpoints $a - R = 0 - 1 = -1$ and $a + R = 0 + 1 = 1$?

For $x = 1$: We get $\sum_{n=0}^{\infty} \frac{1}{3n^2+1}$. Since $0 < \frac{1}{3n^2+1} < \frac{1}{n^2}$ and since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), then by direct comparison $\sum_{n=0}^{\infty} \frac{1}{3n^2+1}$ converges.

For $x = -1$: We get $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^2+1}$. However, we just saw that $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{3n^2+1}\right| = \sum_{n=0}^{\infty} \frac{1}{3n^2+1}$ converges. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^2+1}$ converges by the absolute convergence test.

The interval of convergence is $[-1, 1]$ and includes both endpoints.



Interval of convergence: $[-1, 1]$

EXAMPLE 15.1.10. Determine the radius and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{4^n x^{2n}}{n+1}$.

Solution. We know that the series converges at its center $a = 0$. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4^{n+1}x^{2(n+1)}}{n+2} \cdot \frac{n+1}{4^n x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{4(n+1)}{n+2} \cdot x^2\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{4n}{n} \cdot x^2\right| = |4x^2|.$$

By the ratio test, the series *converges* if

$$|4x^2| < 1 \iff |x^2| < \frac{1}{4} \iff |x| < \frac{1}{2}$$

and *diverges* when $|x| > \frac{1}{2}$. The radius of convergence is $R = \frac{1}{2}$. What about the endpoints $a - R = 0 - \frac{1}{2} = -\frac{1}{2}$ and $a + R = 0 + \frac{1}{2} = \frac{1}{2}$?

For $x = \frac{1}{2}$: We get

$$\sum_{n=0}^{\infty} \frac{4^n(\frac{1}{2})^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

Using the limit comparison test with $\sum_{n=0}^{\infty} \frac{1}{n}$ we see that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} 1.$$

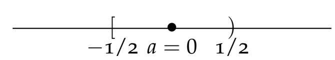
Since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1$), then by the limit comparison test $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

For $x = -\frac{1}{2}$: We get

$$\sum_{n=0}^{\infty} \frac{4^n(-\frac{1}{2})^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

This is an alternating series with $a_n = \frac{1}{n+1} > 0$ where $\lim_{n\to\infty} \frac{1}{n+1} = 0$ and $a_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = a_n$. So by the alternating series test, $\sum_{n=0}^{\infty} \frac{4^n(-\frac{1}{2})^{2n}}{n+1}$ converges.

The interval of convergence is $[-\frac{1}{2}, \frac{1}{2})$ and includes just one of the endpoints.



Interval of convergence: $[-\frac{1}{2}, \frac{1}{2})$

EXAMPLE 15.1.11. Determine the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \cdots 3n(x+2)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

Solution. We know that the series converges at its center $a = -2$. For any $x \neq -2$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3\cdot 6\cdot 9\cdots 3n\cdot(3n+3)(x+2)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{3\cdot 6\cdot 9\cdots 3n(x+2)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{3n+3}{2n+1}\cdot(x+2)\right|$$
$$\overset{\text{HPwrs}}{=} \lim_{n\to\infty}\left|\frac{3n}{2n}\cdot(x+2)\right|$$
$$= \left|\frac{3(x+2)}{2}\right|.$$

By the ratio test, the series *converges* if

$$\left|\frac{3(x+2)}{2}\right| < 1 \iff |x+2| < \frac{2}{3}$$

and *diverges* when $|x+2| > \frac{2}{3}$. The radius of convergence is $R = \frac{2}{3}$.

Bonus: What about the endpoints $a - R = -2 - \frac{2}{3} = -\frac{8}{3}$ and $a + R = -2 + \frac{2}{3} = -\frac{4}{3}$? Determine whether the series converges at either endpoint.

EXAMPLE 15.1.12. Determine the radius and interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty}\frac{n(x+1)^{2n}}{9^n}$.

Solution. We know that the series converges at its center $a = -1$. For any $x \neq -1$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)^{2(n+1)}}{9^{n+1}}\cdot\frac{9^n}{n(x+1)^{2n}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)}{9n}\cdot(x+1)^2\right|$$
$$\overset{\text{HPwrs}}{=} \lim_{n\to\infty}\left|\frac{n}{9n}\cdot(x+1)^2\right|$$
$$= \left|\frac{(x+1)^2}{9}\right|.$$

By the ratio test, the series *converges* if

$$\left|\frac{(x+1)^2}{9}\right| < 1 \iff |(x+1)^2| < 9 \iff |x+1| < 3$$

and *diverges* when $|x+1| > 3$. The radius of convergence is $R = 3$. What about the endpoints $a - R = -1 - 3 = -4$ and $a + R = -1 + 3 = 2$?

For $x = 2$: We get

$$\sum_{n=0}^{\infty}\frac{n(2+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{n3^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{n(3^2)^n}{9^n} = \sum_{n=0}^{\infty}\frac{n9^n}{9^n} = \sum_{n=0}^{\infty} n.$$

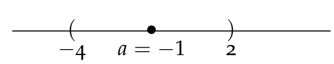
Since $\lim_{n\to\infty} n = \infty \neq 0$, by the nth term test the series diverges.

For $x = -4$: We again get

$$\sum_{n=0}^{\infty}\frac{n(-4+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{n(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{n9^n}{9^n} = \sum_{n=0}^{\infty} n$$

and so the series diverges.

The interval of convergence is $(-4, 2)$ and includes no endpoints.



Interval of convergence: $(-4, 2)$

Practice for radius of convergence only: WEBWORK: Click to try Problems 198 through 207. Use GUEST login, if not in my course. Practice for interval of convergence: WEBWORK: Click to try Problems 208 through 218. Use GUEST login, if not in my course.