Introduction

Calculus I

Calculus I had as its theme "the slope problem." How do we make sense of the notion of slope for curves. . . when we only know what the slope of a line means? The answer, of course, was the to define and determine the derivative of the curve (function).

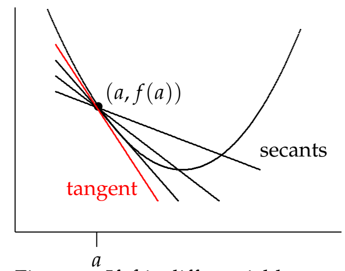


Figure 1: If f is differentiable at a, then then the secant lines through the points $(a, f(a))$ and $(a+h, f(a+h))$ approach the tangent line at $(a, f(a))$.

The reason that this purely geometric question is interesting and important (for non-mathematicians) is that we can interpret many other questions as "slope problems." For example, velocity, acceleration, marginal cost, marginal profit, or any rate of change is really a slope question in disguise.

Additionally points along a curve where the slope is 0 are *critical* because they are potential extreme points for the curve, places where the function obtains its maximum or minimum values. This can be applied to a whole host of problems. For example, what tuition should HWS charge you to maximize its revenue?

Calculus II: The Area Problem

Calculus II has its own theme which also consists of a geometric question. In its simplest form we can state it this way:

The Area Problem. Let f be a continuous (non-negative) function on the closed interval $[a, b]$. Find the area bounded above by $f(x)$, below by the x-axis, and by the vertical lines $x = a$ and $x = b$.

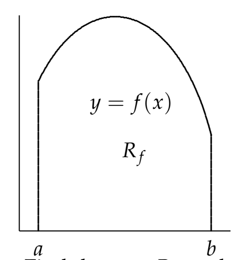


Figure 2: Find the area R_f under a nonnegative continuous curve on the interval $[a, b]$.

For example, if we can solve the general area problem, we will be able to find the area of a circle. Yes, we know it is πr^2. But *why* is it πr^2?! Eventually we will be able to answer this question.

While the area problem is interesting in its own right, it is also important because we can interpret a number of problems as "area problems" in disguise. Other geometrical quantities such as volume or length of a curve can be interpreted as area problems. Problems such as distance travelled or work done (in the physics sense), and calculation of probabilities can also be interpreted as questions about areas. We'll also see that finding the average (or mean) value of a continuous function (like temperature) depends on solving an area problem.

We will start to answer these questions in a couple of classes. For now, we need to quickly review the prerequisites for the course.

Preliminaries and Prerequisites

You should be familiar with all the material in this chapter (Chapter 0) from your previous calculus experience. We will not spend any substantial amount of time reviewing this material.

Note: There is a lot more from Calculus I that I assume you know (e.g., how to graph functions and find their extrema). The material in Chapter 0 is not the only material that I expect you to know.

0.1 *The Derivative*

The derivative is the central concept of Calculus I. You should be very familiar with both the derivative rules and the definition of the derivative.

The Definition of the Derivative

DEFINITION 0.1.1. Let f be a function defined in an open interval containing x. Then the **derivative** $f'(x)$ is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists. When the limit does exist, we say that f **is differentiable at** x.

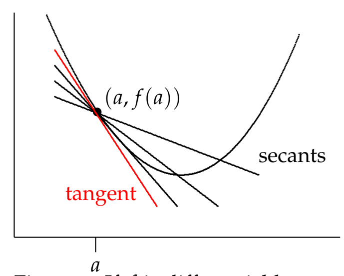


Figure 3: If f is differentiable at a, then then the secant lines through the points $(a, f(a))$ and $(a+h, f(a+h))$ approach the tangent line at $(a, f(a))$.

You may be familiar with an alternative definition for the derivative at a point a which uses slightly different notation

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The derivative represents an *instantaneous rate of change*. Geometrically the derivative can be interpreted as the slope of the function at the point in question.

Derivative Formulæ

These basic derivative expressions should be familiar. In the last few, assume both f and g are differentiable functions and that a is a constant.

Differentiation Formulæ

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(kx) = k$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

$\frac{d}{dx}(\sin ax) = a\cos ax$

$\frac{d}{dx}(\cos ax) = -a\sin ax$

$\frac{d}{dx}(\tan ax) = a\sec^2 ax$

$\frac{d}{dx}(\sec ax) = a\sec ax\tan ax$

$\frac{d}{dx}(e^{ax}) = e^{ax}$

$\frac{d}{dx}(\arcsin\frac{x}{a}) = \frac{1}{\sqrt{a^2-x^2}}$

$\frac{d}{dx}(\arctan\frac{x}{a}) = \frac{a}{a^2+x^2}$

$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

$\frac{d}{dx}[cf(x)] = cf'(x)$

$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Stop! Review

YOU TRY IT 0.1. The theme of Calculus I was differentiation. State the derivatives of each of these functions.

(*a*) x^n (*b*) $\sin x$ (*c*) $\cos x$ (*d*) $\tan x$ (*e*) $\sec x$ (*f*) c

(*g*) $2e^x$ (*h*) $5 + 6\ln x$ (*i*) x^{-n} (*j*) $\sqrt{x}$ (*k*) $\sqrt[n]{x}$ (*l*) $\sqrt[n]{x^m}$

(*m*) $x^6\sin x$ (*n*) $\cos(6x) + 1$ (*o*) $e^x\tan(4x)$ (*p*) $7e^{\sec x}$

(*q*) $\ln(2x^4)$ (*r*) $\arctan(x^2)$ (*s*) $\frac{x^2+1}{e^{2x}}$ (*t*) $\sin^2 x$

ANSWER TO YOU TRY IT 0.1.

(*a*) nx^{n-1} (*b*) $\cos x$ (*c*) $-\sin x$ (*d*) $\sec^2 x$ (*e*) $\sec x\tan x$ (*f*) 0

(*g*) $2e^x$ (*h*) $6x^{-1}$ (*i*) $-nx^{-n-1}$ (*j*) $\frac{1}{2}x^{-1/2}$ (*k*) $\frac{1}{n}x^{\frac{1-n}{n}}$ (*l*) $\frac{m}{n}x^{\frac{m-n}{n}}$

(*m*) $6x^5\sin x + x^6\cos x$ (*n*) $-6\sin(6x)$ (*o*) $e^x\tan(4x) + e^x(4\sec^2 4x)$

(*p*) $7e^{\sec x}\sec x\tan x$ (*q*) $\frac{4}{x}$ (*r*) $\frac{2x}{1+x^4}$ (*s*) $\frac{2xe^{2x} - (x^2+1)2e^{2x}}{e^{4x}} = \frac{2x - 2(x^2+1)}{e^{2x}}$

(*t*) $2\sin x\cos x$

YOU TRY IT 0.2. State the derivatives of each of these functions.

(*a*) $f(x) = \cos^4 x$ (*b*) $g(x) = xe^{\sin x}$ (*c*) $g(x) = \ln(6x^4)$

ANSWER TO YOU TRY IT 0.2.

(*a*) $f'(x) = -4\sin x\cos^3 x$ (*b*) $g'(x) = (1 + x\cos x)e^{\sin x}$ (*c*) $g'(x) = \frac{1}{6x^4}\cdot 24x^3 = 4x^{-1}$

WEBWORK: Click to try Problems 1 through 5. Use GUEST login, if not in my course.

0.2 The Mean Value Theorem

The Mean Value Theorem is one of the most important theorems in elementary calculus. It relates the global behavior of a function (how it changes over an entire interval) to the local behavior of a function (the derivative of the function at a particular point). The MVT is used to prove a number of important results in calculus. For example, it is used to prove the first derivative test: If $f'(x) > 0$, then f is increasing.

You should be able to state the MVT and draw a graph that illustrates it.

THEOREM 0.2.1 (MVT: The Mean Value Theorem)**.** Assume that

1. f is continuous on the closed interval $[a, b]$;
2. f is differentiable on the open interval (a, b).

Then there is some point c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is equivalent to saying $f(b) - f(a) = f'(c)(b - a)$.

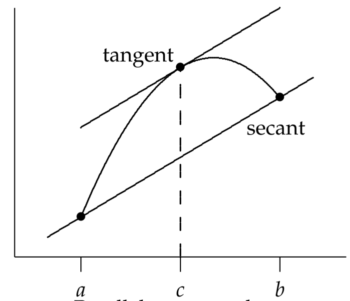


Figure 4: Parallel secant and tangent lines exist when the Mean Value Theorem applies.

What the MVT Says. Let's interpret the MVT geometrically. The *average* or *mean rate of change* of f on the interval $[a, b]$ is just

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

This is the familiar *secant slope* of the line through $(a, f(a))$ and $(b, f(b))$. The MVT says if f is differentiable, we can find a point c between a and b so that the *instantaneous rate of change* or *tangent slope* or the *derivative* $f'(c)$ is the same as the secant slope. When two lines (here the secant and tangent) have the same slope, they are parallel. (See Figure 4 above.) If we think of the derivative as 'velocity', then the MVT says that if you calculate your average velocity over a time interval, then *at some point during the interval your instantaneous velocity actually equals your average velocity*.

Mostly the MVT gets used to prove other theorems. But we can look at an example or two to see how it works.

EXAMPLE 0.2.2. Show how the MVT applies to $f(x) = x^3 - 6x + 1$ on $[0, 3]$.

Solution. Check the two conditions (hypotheses)

1. f is continuous on the closed interval $[0, 3]$ because it is a polynomial;
2. f is differentiable on the open interval $(0, 3)$ again because it is a polynomial;

So the MVT applies: There is some point c between 0 and 3 so that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{10 - 1}{3} = 3.$$

Since $f'(x) = 3x^2 - 6$, then

$$f'(c) = 3c^2 - 6 = 3 \Rightarrow 3c^2 = 9 \Rightarrow c = \pm\sqrt{3}$$

Only $c = \sqrt{3}$ is in the interval, so this is the value of c.

EXAMPLE 0.2.3. Show there does not exist a differentiable function on $[1, 5]$ with $f(1) = -3$ and $f(5) = 9$ with $f'(x) \leq 2$ for all x.

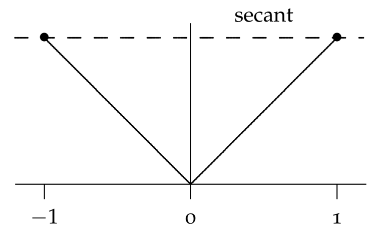
Solution. The MVT would apply to such a function f: So there should be some point c between 1 and 5 so that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{9 - (-3)}{4} = 3.$$

But supposedly $f'(x) \leq 2$ for all x. Contradiction. So no such f can exist.

EXAMPLE 0.2.4. It is also important to understand why the hypotheses of the theorem are necessary. Take the function $f(x) = |x|$ on the interval $[-1, 1]$. Notice that if we tried to apply the MVT here, the endpoints would be $a = -1$ and $b = 1$. So there should be a point c between -1 and 1 so that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{|1| - |-1|}{1 + 1} = \frac{1 - 1}{2} = 0.$$



But you can see from the graph of $f(x) = |x|$ that the slope is never 0; $f'(x)$ is either -1 when $x < 0$ or 1 when $x > 0$. The problem here is that $f(x) = |x|$ is NOT differentiable at $x = 0$. $|x|$ *fails to satisfy the hypotheses* of the MVT and for that reason $|x|$ does not satisfy the conclusion of the MVT on the interval $[-1, 1]$.

Using the MVT. We will almost never use the MVT as in the three examples above. That does not mean you should not spend time understanding these examples because they illustrate the key ideas of the theorem. However, the key value of the MVT is in proving other results. We mentioned earlier that the MVT is used to prove that functions with positive (negative) derivatives are increasing (resp., decreasing). We will see a further application of the MVT when we we prove the Fundamental Theorem of Calculus relating antidifferentiation to the problem of finding the area under a curve.

0.3 *Antiderivatives*

If you know the velocity of an object, can you determine the position of the object? This could happen in a car, say, where the speedometer readings were being recorded. Can the position of the car be determined from this information? Similarly, can the position of an airplane be determined from the black box which records the airspeed?

Remembering that velocity is really just a derivative, we can ask this same question more generally: Given $f'(x)$ can we find the function $f(x)$? We usually state the problem this way.

DEFINITION 0.3.1. Let $f(x)$ be a function defined on an interval I. We say that $F(x)$ is an **antiderivative** of $f(x)$ on I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

EXAMPLE 0.3.2. If $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative of f because

$$F'(x) = 2x = f(x).$$

But so is $G(x) = x^2 + 1$ or, more generally, $H(x) = x^2 + c$.

Are there 'other' antiderivatives of $f(x) = 2x$ besides those of the form $H(x) = x^2 + c$? We can use the MVT to show that the answer is 'No.' The proof will require two steps.

THEOREM 0.3.3. If $F'(x) = 0$ for all x in an interval I, then $F(x) = k$ is a constant function.

This makes a lot of sense: If the velocity of an object is 0, then its position is constant (not changing). Here's the

Proof. To show that $F(x)$ is constant, we must show that any two output values of F are the same, i.e., $F(a) = F(b)$ for all a and b in I.

So pick any a and b in I (with $a < b$). Because F is differentiable on I, then F is both continuous and differentiable on the smaller interval $[a, b]$. So the MVT applies (Theorem 0.2.1—look at its equivalent statement). There is a point c between a and b so that

$$F(b) - F(a) = F'(c)(b - a)$$

because we are given that F' is always 0 we can substitute in and say

$$F(b) - F(a) = 0(b - a) = 0$$

Since $F(b) - F(a) = 0$, this means $F(b) = F(a)$. In other words, F is constant. □

THEOREM 0.3.4. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I, then $G(x) = F(x) + k$. That is, any two antiderivatives of the same function differ by a constant.

Proof. Assume $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I. Consider the function $G(x) - F(x)$ on I. Then

$$\frac{d}{dx}[G(x) - F(x)] = G'(x) - F'(x) = f(x) - f(x) = 0.$$

Therefore, by Theorem 0.3.3 that we just proved

$$G(x) - F(x) = k$$

so

$$G(x) = F(x) + k.$$

□

DEFINITION 0.3.5. If $F(x)$ is any antiderivative of $f(x)$, we say that $F(x) + c$ is the **general antiderivative** of $f(x)$ on I.

Notation for Antiderivatives

Antidifferentiation is also called 'indefinite integration.'

$$\int f(x)\,dx = F(x) + c.$$

- $\int$ is the integration symbol
- $f(x)$ is called the integrand
- dx indicates the variable of integration (here it is x)
- $F(x)$ is a particular antiderivative of $f(x)$
- and c is the constant of integration.
- We refer to $\int f(x)\,dx$ as an 'antiderivative of $f(x)$' or an 'indefinite integral of f.'

  Here are several examples.

$$\int \cos t\,dt = \sin t + c$$

$$\int e^z\,dz = e^z + c$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + c$$

Antidifferentiation reverses differentiation so

$$\int F'(x)\,dx = F(x) + c$$

as long as $F'(x)$ is continuous. And differentiation undoes antidifferentiation so

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Since differentiation and antidifferentiation are reverse processes, each derivative rule has a corresponding antidifferentiation rule.

Differentiation	**Antidifferentiation**
$\frac{d}{dx}(c) = 0$	$\int 0\,dx = c$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\|x\| + c$
$\frac{d}{dx}(\sin ax) = a\cos ax$	$\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
$\frac{d}{dx}(\cos ax) = -a\sin ax$	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
$\frac{d}{dx}(\tan ax) = a\sec^2 ax$	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$
$\frac{d}{dx}(\sec ax) = \sec ax \tan ax$	$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}(\arcsin \frac{x}{a}) = \frac{1}{\sqrt{a^2-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin \frac{x}{a} + c$
$\frac{d}{dx}(\arctan \frac{x}{a}) = \frac{a}{a^2+x^2}$	$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan \frac{x}{a} + c$

0.4 *General Antiderivative Rules*

The key idea is that each derivative rule can be written as an antiderivative rule. We've seen how this works with specific functions like $\sin x$ and e^x and now we examine how the general derivative rules can be 'reversed.'

THEOREM 0.4.1 (Sum Rule)**.** The sum rule for derivatives says

$$\frac{d}{dx}(F(x) \pm G(x)) = \frac{d}{dx}(F(x)) \pm \frac{d}{dx}(G(x)).$$

The corresponding antiderivative rule is

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

THEOREM 0.4.2 (Constant Multiple Rule)**.** The constant multiple rule for derivatives says

$$\frac{d}{dx}(cF(x)) = c\frac{d}{dx}(F(x)).$$

The corresponding antiderivative rule is

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

Simple Examples

Be sure you understand how the basic antiderivative rules apply in each of these problems.

EXAMPLE 0.4.3. Each of these antiderivatives uses multiple rules. Try to identify them.

$$\int 8x^3 - 7\sqrt{x}\,dx = \int 8x^3\,dx - \int 7x^{1/2}\,dx = 8\int x^3\,dx - 7\int x^{1/2}\,dx$$
$$= \frac{8x^4}{4} - \frac{7x^{3/2}}{3/2} + c = 2x^4 - \frac{14x^{3/2}}{3} + c$$

$$\int 6\cos 2x - \frac{7}{x} + 2x^{-1/3}\,dx = 6\int \cos 2x\,dx - 7\int \frac{1}{x}\,dx + 2\int x^{-1/3}\,dx$$
$$= 6\cdot\frac{1}{2}\cdot\sin 2x - 7\ln|x| + \frac{2x^{2/3}}{2/3} + c = 3\sin 2x - 7\ln|x| + 3x^{2/3} + c.$$

$$\int 3e^{x/2} - \frac{8}{\sqrt{16-x^2}}\,dx = 3\int e^{\frac{1}{2}x}\,dx - 8\int \frac{1}{\sqrt{4^2-x^2}}\,dx = 3\cdot\frac{1}{\frac{1}{2}}\cdot e^{x/2} - 8\arcsin\frac{x}{4} + c$$
$$= 6e^{x/2} - 8\arcsin\frac{x}{4} + c.$$

YOU TRY IT 0.3. Determine each of the antiderivatives of each of the functions below.

$(a)\ \int \sqrt{x} + x^{99}\,dx \qquad (b)\ \int 5x^{1/4} - 3e^{-2x}\,dx \qquad (c)\ \int 4\sec^2\frac{x}{3} + 3x^{-2}\,dx$

$(a)\ \frac{2}{3}x^{3/2} + \frac{1}{100}x^{100} + c \qquad (b)\ 4x^{5/4} + \frac{3}{2}e^{-2x} + c \qquad (c)\ 12\tan\frac{x}{3} - 3x^{-1} + c$

ANSWER TO YOU TRY IT 0.3.

Examples with Rewriting

Sometimes the antidifferentiation process is greatly simplified by rewriting the integrand before any antidifferentiation is attempted. This may involve rewriting powers as exponents, dividing out common factors, or multiplying out products.

Integral		**Rewritten**		**sol**
$\int 4\sqrt[5]{t^2}\,dt$	=	$4\int t^{2/5}\,dt$	=	$\frac{20}{7}t^{7/5} + c$
$\int \frac{x^4+2}{x^2}\,dx$	=	$\int x^2 + 2x^{-2}\,dx$	=	$\frac{1}{3}x^3 - 2x^{-1} + c$
$\int 6x^2(x^4-1)\,dx$	=	$\int 6x^6 - 6x^2\,dx$	=	$\frac{6}{7}x^7 - 2x^3 + c$
$\int \frac{1}{6x^4}\,dx$	=	$\frac{1}{6}\int x^{-4}\,dx$	=	$-\frac{1}{18}x^{-3} + c$
$\int \frac{7}{4\sqrt[3]{s}}\,ds$	=	$\frac{7}{4}\int s^{-1/3}\,ds$	=	$\frac{21}{8}s^{2/3} + c$
$\int \frac{4}{9+x^2}\,dx$	=	$4\int \frac{1}{3^2+x^2}\,dx$	=	$4\cdot\frac{1}{3}\cdot\arctan\frac{x}{3} + c = \frac{4}{3}\arctan\frac{x}{3} + c$
$\int \frac{1}{2\sqrt{25-x^2}}\,dx$	=	$\frac{1}{2}\int \frac{1}{\sqrt{5^2-x^2}}\,dx$	=	$\frac{1}{2}\arcsin\frac{x}{5} + c$
$\int \frac{2\sqrt{x}+3}{x}\,dx$	=	$2\int x^{-1/2}\,dx + 3\int \frac{1}{x}\,dx$	=	$4x^{1/2} + 3\ln|x| + c$

YOU TRY IT 0.4. Determine these antiderivatives.

$(a)\ \int 2\sec 3x\tan 3x\,dx \qquad (b)\ \int \frac{7x^3 - \sqrt{x} - 4}{x}\,dx \qquad (c)\ \int 5\sqrt[4]{x^3} + 3x^{-2}\,dx$

$(a)\ \frac{2}{3}\sec 3x + c \qquad (b)\ \frac{7x^3}{3} - 2x^{1/2} - 4\ln|x| + c \qquad (c)\ \frac{20x^{7/4}}{7} - 3x^{-1} + c$

ANSWER TO YOU TRY IT 0.4.

WEBWORK: Click to try Problems 6 through 9. Use GUEST login, if not in my course.

YOU TRY IT 0.5. Which general rules have we not yet reversed? Try to write down the corresponding antiderivative rules similar to Theorem 0.4.1 and Theorem 0.4.2.

0.5 Evaluating 'c' (Initial Value Problems)

In each general antiderivative there is an unknown constant c. This is what makes the integral "indefinite." If an object is moving along a straight line, we know that the derivative of the position function $s(t)$ is the velocity function $s'(t) = v(t)$. What this means is if we know the velocity $v(t)$ of the car we are driving in (just look at the speedometer), we can determine the position $s(t)$ of the car, up to a constant c *if we can find an antiderivative* for $v(t)$. If we have more information, the position of the car at a particular time say, then we are able to determine the precise antiderivative and exact position. Let's see how this works in general before we apply it to motion problems.

EXAMPLE 0.5.1. Suppose that $f'(x) = e^x + 2x$ and $f(0) = 3$. Find $f(x)$. In this context $f(0)$ is sometimes called the *initial value* and such questions are referred to as *initial value problems*. [How could you interpret this information in terms of motion of a car?]

Solution. $f(x)$ must be an antiderivative of $f'(x)$ so

$$f(x) = \int f'(x)\,dx = \int e^x + 2x\,dx = e^x + x^2 + c.$$

Now use the initial value to solve for c:

$$f(0) = e^0 + 0^2 + c = 3 \Rightarrow 1 + c = 3 \Rightarrow c = 2.$$

Therefore, $f(x) = e^x + x^2 + 2$.

EXAMPLE 0.5.2. Suppose that $f'(x) = 6x^2 - 2x^3$ and $f(1) = 4$. Find $f(x)$.

Solution. Again $f(x)$ must be an antiderivative of $f'(x)$ so

$$f(x) = \int f'(x)\,dx = \int 6x^2 - 2x^3\,dx = 2x^3 - \frac{x^4}{2} + c.$$

Now use the 'initial' value to solve for c:

$$f(1) = 2 - 1/2 + c = 4 \Rightarrow c = 5/2.$$

Therefore, $f(x) = 2x^3 - \frac{1}{2}x^4 + \frac{5}{2}$.

EXAMPLE 0.5.3. Suppose that $f''(t) = 6t^{-2}$ (think acceleration) with $f'(1) = 8$ (think velocity) and $f(1) = 3$ (think position). Find $f(t)$.

Solution. First find $f'(t)$ which is just the antiderivative of $f''(t)$. So

$$f'(t) = \int f''(t)\,dt = \int 6t^{-2}\,dt = -6t^{-1} + c.$$

Now use the 'initial' value for $f'(t)$ to solve for c:

$$f'(1) = -6(1) + c = 8 \Rightarrow c = 14.$$

Therefore, $f'(t) = -6t^{-1} + 14$. Now we are back to the earlier problem.

$$f(t) = \int f'(t)\,dt = \int -6t^{-1} + 14\,dt = -6\ln|t| + 14t + c.$$

Now use the 'initial' value of f to solve for c:

$$f(1) = -6\ln 1 + 14(1) + c = 3 \Rightarrow 6(0) + 14 + c = 3 \Rightarrow c = -11.$$

So $f(t) = 6\ln|t| + 14t - 11$.

More Practice

EXAMPLE 0.5.4. Find f given that $f'(x) = 6\sqrt{x} + 5x^{\frac{3}{2}}$ where $f(1) = 10$.

Solution. $f(x)$ must be an antiderivative of $f'(x)$ so

$$f(x) = \int f'(x)\,dx = \int 6\sqrt{x} + 5x^{\frac{3}{2}}\,dx = 4x^{3/2} + 2x^{5/2} + c.$$

Use the 'initial' value to solve for c:

$$f(1) = 4 + 2 + c = 10 \Rightarrow c = 4.$$

Therefore, $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

EXAMPLE 0.5.5. Find f given that $f''(\theta) = \sin\theta + \cos\theta$ where $f'(0) = 1$ and $f(0) = 2$.

Solution. First find $f'(\theta)$ which must be the antiderivative of $f''(\theta)$. So

$$f'(\theta) = \int f''(\theta)\,d\theta = \int \sin\theta + \cos\theta\,d\theta = -\cos\theta + \sin\theta + c.$$

Now use the initial value for $f'(\theta)$ to solve for c:

$$f'(0) = -\cos 0 + \sin 0 + c = -1 + 0 + c = 1 \Rightarrow c = 2.$$

Therefore, $f'(\theta) = -\cos\theta + \sin\theta + 2$.

$$f(\theta) = \int f'(\theta)\,d\theta = \int -\cos\theta + \sin\theta + 2\,d\theta = -\sin\theta - \cos\theta + 2\theta + c.$$

Now use the initial value of f to solve for c:

$$f(0) = -\sin 0 - \cos 0 + 2(0) + c = 0 - 1 + c = 2 \Rightarrow c = 3.$$

So $f(\theta) = -\sin\theta - \cos\theta + 2\theta + 3$.

0.6 *Motion Problems*

In Calculus I you interpreted the first and second derivatives as velocity and acceleration in the context of motion. So let's apply the initial value problem results to motion problems. Recall that

- $s(t) =$ position at time t.
- $s'(t) = v(t) =$ velocity at time t.
- $s''(t) = v'(t) = a(t) =$ acceleration at time t.

Therefore

- $\int a(t)\,dt = v(t) + c_1 =$ velocity.
- $\int v(t)\,dt = s(t) + c_2 =$ position at time t.

We will need to use additional information to evaluate the constants c_1 and c_2.

EXAMPLE 0.6.1. Suppose that the acceleration of an object is given by $a(t) = 2 - \cos t$ for $t \geq 0$ with

- $v(0) = 1$, this is also denoted v_0
- $s(0) = 3$, this is also denoted s_0.

Find $s(t)$.

Solution. First find $v(t)$ which is the antiderivative of $a(t)$.

$$v(t) = \int a(t)\,dt = \int 2 - \cos t\,dt = 2t - \sin t + c_1.$$

Now use the initial value for $v(t)$ to solve for c_1:

$$v(0) = 0 - 0 + c_1 = 1 \Rightarrow c_1 = 1.$$

Therefore, $v(t) = 2t - \sin t + 1$. Now solve for $s(t)$ by taking the antiderivative of $v(t)$.

$$s(t) = \int v(t)\,dt = \int 2t - \sin t + 1\,dt = t^2 + \cos t + t + c_2$$

Now use the initial value of s to solve for c_2:

$$s(0) = 0 + \cos 0 + c_2 = 3 \Rightarrow 1 + c_2 = 3 \Rightarrow c_2 = 2.$$

So $s(t) = t^2 + \cos t + 2t + 2$.

EXAMPLE 0.6.2. If acceleration is given by $a(t) = 10 + 3t - 3t^2$, find the exact position function if $s(0) = 1$ and $s(2) = 11$.

Solution. First

$$v(t) = \int a(t)\,dt = \int 10 + 3t - 3t^2\,dt = 10t + \tfrac{3}{2}t^2 - t^3 + c.$$

Now

$$s(t) = \int 10t + \tfrac{3}{2}t^2 - t^3 + c\,dt = 5t^2 + \tfrac{1}{2}t^3 - \tfrac{1}{4}t^4 + ct + d.$$

But $s(0) = 0 + 0 - 0 + 0 + d = 1$ so $d = 1$. Then $s(2) = 20 + 4 - 4 + 2c + 1 = 11$ so $2c = -10 \Rightarrow c = -5$. Thus, $s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 - 5t + 1$.

EXAMPLE 0.6.3. If acceleration is given by $a(t) = \sin t + \cos t$, find the position function if $s(0) = 1$ and $s(2\pi) = -1$.

Solution. First

$$v(t) = \int a(t)\,dt = \int \sin t + \cos t\,dt = -\cos t + \sin t + c.$$

Now

$$s(t) = \int -\cos t + \sin t + c\,dt = -\sin t - \cos t + ct + d.$$

But $s(0) = 0 - 1 + 0 + 0 + d = 1$ so $d = 2$. Then $s(2\pi) = 0 - 1 + 2c\pi + 2 = -1$ so $2\pi c = -2 \Rightarrow c = -\frac{1}{\pi} + 2$. Thus, $s(t) = \cos t + \sin t - \frac{1}{\pi}t$.

0.7 Constant Acceleration: Gravity

In many motion problems the acceleration is constant. This happens when an object is thrown or dropped and the only acceleration is due to gravity. In such a situation we have

- $a(t) = a$, constant acceleration
- with initial velocity $v(0) = v_0$
- and initial position $s(0) = s_0$.

Then

$$v(t) = \int a(t)\,dt = \int a\,dt = at + c.$$

But

$$v(0) = a \cdot 0 + c = v_0 \Rightarrow c = v_0.$$

So

$$\boxed{v(t) = at + v_0.}$$

Next,

$$s(t) = \int v(t)\,dt = \int at + v_0\,dt = \tfrac{1}{2}at^2 + v_0 t + c.$$

At time $t = 0$,

$$s(0) = \tfrac{1}{2}a(0)^2 + v_0(0) + c = s_0 \Rightarrow c = s_0.$$

Therefore

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0.}$$

EXAMPLE 0.7.1. Suppose a ball is thrown with initial velocity 96 ft/s from a roof top 432 feet high. The acceleration due to gravity is constant $a(t) = -32$ ft/s^2. Find $v(t)$ and $s(t)$. Then find the maximum height of the ball and the time when the ball hits the ground.

Solution. Recognizing that $v_0 = 96$ and $s_0 = 432$ and that the acceleration is constant, we may use the general formulas we just developed.

$$\boxed{v(t) = at + v_0 = -32t + 96}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + 96t + 432.}$$

The max height occurs when the velocity is 0 (when the ball stops rising):

$$v(t) = -32t + 96 = 0 \Rightarrow t = 3 \Rightarrow s(3) = -144 + 288 + 432 = 576 \text{ ft}.$$

The ball hits the ground when $s(t) = 0$.

$$s(t) = -16t^2 + 96t + 432 = -16(t^2 - 6t - 27) = -16(t-9)(t+3) = 0.$$

So $t = 9$ only (since $t = -3$ does not make sense).

EXAMPLE 0.7.2. A person drops a stone from a bridge. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

Solution. Here's what we know. $v_0 = 0$ (dropped) and $s(5) = 0$ (hits water). And we know acceleration is constant, $a = -32$ ft/s^2. We want to find the height of the bridge, which is just s_0. Use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = -32t}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + s_0.}$$

Now we use the position we know: $s(5) = 0$.

$$s(5) = -16(5)^2 + s_0 \Rightarrow s_0 = 400 \text{ ft}.$$

Notice that we did not need to use the velocity function.

YOU TRY IT 0.6 (Extra Credit). In the previous problem we did not take into account that sound does not travel instantaneously in your calculation above. Assume that sound travels at 1120 ft/s. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

Check on your answer: Should the bridge be higher or lower than in the preceding example? Why?

EXAMPLE 0.7.3. Here's a variation. This time we will use metric units. Suppose a ball is thrown with unknown initial velocity v_0 m/s from a roof top 49 meters high and the position of the ball at time $t = 3$ is $s(3) = 0$. The acceleration due to gravity is constant $a(t) = -9.8 \text{ m/s}^2$. Find $v(t)$ and $s(t)$.

Solution. This time v_0 is unknown but $s_0 = 49$ and $s(3) = 0$. Again the acceleration is constant so we may use the general formulas for this situation.

$$v(t) = at + v_0 = -9.8t + v_0$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = -4.9t^2 + v_0t + 49.$$

But we know that

$$s(3) = -4.9(3)^2 + v_0 \cdot 3 + 49 = 0$$

which means

$$3v_0 = 4.9(9) - 4.9(10) = -4.9 \Rightarrow v_0 = -4.9/3.$$

So

$$\boxed{v(t) = -9.8t - \tfrac{49}{30}}$$

and

$$\boxed{s(t) = -4.9t^2 - \tfrac{49}{30}t + 49.}$$

Interpret $v_0 = -4.9/3$.

EXAMPLE 0.7.4. Mo Green is attempting to run the 100m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his *acceleration* is constant, a, over the entire race. Determine his velocity function. (a will still appear as an unknown constant.) Determine his position function. There should be no unknown constants in your equation at this point. What is his velocity at the end of the race? Do you think this is realistic?

Solution. We have: constant acceleration $= a \text{ m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = \tfrac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2} = 2.0825 \text{ m/s}^2$. So $s(t) = 2.0825t^2$. Mo's velocity at the end of the race is $v(9.8) = a \cdot 9.8 = 2.0825(9.8) = 20.41$ m/s... not realistic.

EXAMPLE 0.7.5. A stone dropped off a cliff hits the ground with speed of 120 ft/s. What was the height of the cliff?

Solution. Notice that $v_0 = 0$ (dropped!) and s_0 is unknown but is equal to the cliff height, and that the acceleration is constant $a = -32$ ft/. Use the general formulas for motion with constant acceleration:

$$\boxed{v(t) = at + v_0 = -32t + 0 = -32t}.$$

Now we use the velocity function and the one velocity value we know: $v = -120$ when it hits the ground. So the *time* when it hits the ground is given by

$$v(t) = -32t = -120 \Rightarrow t = 120/32 = 15/4$$

when it hits the ground. Now remember when it hits the ground the height is 0. So $s(15/4) = 0$. But we know

$$\boxed{s(t) = \frac{1}{2}at^2 + v_0t + s_0 = -16t^2 + 0t + s_0 = -16t^2 + s_0.}$$

Now substitute in $t = 15/4$ and solve for s_0.

$$s(15/4) = 0 \Rightarrow -16(15/4)^2 + s_0 = 0 \Rightarrow s_0 = 15^2 = 225.$$

The cliff height is 225 feet.

EXAMPLE 0.7.6. A car is traveling at 90 km/h when the driver sees a deer 75 m ahead and slams on the brakes. What constant deceleration is required to avoid hitting Bambi? [Note: First convert 90 km/h to m/s.]

Solution. Let's list all that we know. $v_0 = 90$ km/h or $\frac{90000}{60 \cdot 60} = 25$ m/s and $s_0 = 0$. Let time t^* represent the time it takes to stop. Then $s(t^*) = 75$ m. Now the car is stopped at time t^*, so we know $v(t^*) = 0$. Finally we know that acceleration is an unknown constant, a, which is what we want to find.

Now we use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = at + 25}$$

and

$$\boxed{s(t) = \frac{1}{2}at^2 + v_0t + s_0 = \frac{1}{2}at^2 + 25t.}$$

Now use the other velocity and position we know: $v(t^*) = 0$ and $s(t^*) = 75$ when the car stops. So

$$v(t^*) = at^* + 25 = 0 \Rightarrow t^* = -25/a$$

and

$$s(t^*) = \frac{1}{2}a(t^*)^2 + 25t^* = \frac{1}{2}a(-25/a)^2 + 25(-25/a) = 75.$$

Simplify to get

$$\frac{625a}{2a^2} - \frac{625}{a} = \frac{625}{2a} - \frac{1350}{2a} = -\frac{625}{2a} = 75 \Rightarrow 150a = -625$$

so

$$a = -\frac{625}{150} = -\frac{25}{6} \text{ m/s.}$$

(Why is acceleration negative?)

EXAMPLE 0.7.7. One car intends to pass another on a back road. What constant acceleration is required to increase the speed of a car from 30 mph (44 ft/s) to 50 mph ($\frac{220}{3}$ ft/s) in 5 seconds?

Solution. Given: $a(t) = a$ constant. $v_0 = 44$ ft/s. $s_0 = 0$. And $v(5) = \frac{220}{3}$ ft/s. Find a. But

$$v(t) = at + v_0 = at + 44.$$

So

$$v(5) = 5a + 44 = \frac{220}{3} \Rightarrow 5a = \frac{220}{3} - 44 = \frac{88}{3}.$$

Thus $a = \frac{88}{15}$.

WEBWORK: Click to try Problems 10 through 11. Use GUEST login, if not in my course.

YOU TRY IT 0.7. A toy bumper car is moving back and forth along a straight track. Its acceleration is $a(t) = \cos t + \sin t$. Find the particular velocity and position functions given that $v(\pi/4) = 0$ and $s(\pi) = 1$.

ANSWER TO YOU TRY IT 0.7. $v(t) = \int a(t)\,dt = \int \cos t + \sin t\,dt = \sin t - \cos t + c$. So $v(\pi/4) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + c = 0 \Rightarrow c = 0$. Thus, $v(t) = \sin t - \cos t$. Now $s(t) = \int v(t)\,dt = \int \sin t - \cos t\,dt = -\cos t - \sin t + c$. Since $s(\pi) = -(-1) - 0 + c = 1 \Rightarrow c = 0$. So $s(t) = -\cos t - \sin t$.

YOU TRY IT 0.8 (Graphical integration). Let $F(x)$ be an antiderivative of $f(x)$ on $[0,6]$, where f in Figure 5. Since F is an antiderivative of f, then $F' = f$. Use this relationship to answer the following questions.

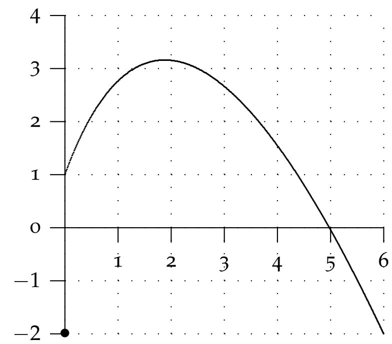


Figure 5: This is the graph of $f(x)$. Can you draw a graph of its antiderivative $F(x)$?

(*a*) On what interval(s) is F increasing? (Give a reason for your answer.)

(*b*) At what point(s), if any, does F have a local max?

(*c*) On what interval(s) is F concave up?

(*d*) At what point(s), if any, does F have points of inflection?

(*e*) Assume that F passes through $(0, -2)$ indicated with a •; draw a potential graph of F.

ANSWER TO YOU TRY IT 0.8. Justify each.

(*a*) $[0,5]$ (*b*) $x = 5$ (*c*) $[0,2]$ (*d*) $x = 2$