1.3 Continuous Functions and Riemann Sums

In Example 1.2.2 we saw that $\lim_{n\to\infty} \text{Lower}(n) = \lim_{n\to\infty} \text{Upper}(n)$ for the function $f(x) = 1 + \frac{1}{2}x^2$ on $[0,2]$. This is no accident. It is an example of the following theorem.

THEOREM 1.3.1. Let f be a (non-negative) continuous function on the closed interval $[a,b]$. The limits as $n \to \infty$ of the upper and lower Riemann sums for regular partitions both exist and are equal. That is,

$$\lim_{n\to\infty} \text{Lower}(n) = \lim_{n\to\infty} \text{Upper}(n),$$

in other words,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(m_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f(M_k)\Delta x.$$

The proof of this theorem is hard. It is covered in our Math 331 course. But you have seen at least one example where this theorem holds and we will see other examples later. The function need not be non-negative, but the answer will represent an area only when the function is non-negative.

There is an important consequence to Theorem 1.3.1. Because $f(m_k)$ and $f(M_k)$ are the minimum and maximum values of f on the kth subinterval,[1] if c_k is *any* number in the kth interval then

$$f(m_k) \le f(c_k) \le f(M_k).$$

So

$$\sum_{k=1}^{n} f(m_k)\Delta x \le \sum_{k=1}^{n} f(c_k)\Delta x \le \sum_{k=1}^{n} f(M_k)\Delta x.$$

In other words,

$$\text{Lower}(n) \le \sum_{k=1}^{n} f(c_k)\Delta x \le \text{Upper}(n).$$

Taking limits

$$\lim_{n\to\infty} \text{Lower}(n) \le \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x \le \lim_{n\to\infty} \text{Upper}(n). \tag{1.6}$$

But Theorem 1.3.1 says the first and last limits are the same, so by the squeeze theorem for limits, all three limits in (1.6) must be equal. That is,

THEOREM 1.3.2. Let f be a (non-negative) continuous function on the closed interval $[a,b]$. Then no matter how we select the sample points c_k in each subinterval,

$$\lim_{n\to\infty} \text{Lower}(n) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \lim_{n\to\infty} \text{Upper}(n).$$

Translation of Theorem 1.3.2. If f is a (non-negative) continuous function on the closed interval $[a,b]$, then *we can use any convenient point* c_k in the kth subinterval to evaluate f... we do not need to choose m_k or M_k where the min or max occurs. Usually the most convenient point to choose is $x_k = a + k\Delta x$ which is the right-hand endpoint of the interval because this usually produces a simple summation formula.

The other consequence of this result is that we can now define the area under a continuous curve. Here we do need the function to be non-negative.

[1] This is why we need f to be continuous. Continuous functions always have both a max and a min on any closed interval.

DEFINITION 1.3.3. Let f be a non-negative, continuous function on the closed interval $[a, b]$. The **area** bounded above by the graph of f, below by the x-axis, on the left by the line $x = a$, and on the right by $x = b$ is given by

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and c_k is any point in the kth subinterval of the regular partition of $[a, b]$ into n subintervals.

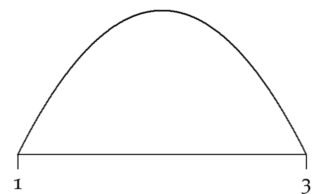


Figure 1.18: The graph of the parabola $f(x) = -x^2 + 4x - 3$ on $[1, 3]$.

EXAMPLE 1.3.4. Use Definition 1.3.3 to determine the area under the curve $y = f(x) = -x^2 + 4x - 3$ on the interval $[1, 3]$.

Solution. Notice in Figure 1.18 that because the $f(x)$ is both increasing and decreasing on the interval, if we computed Upper(n), the max values of f would sometimes occur at the right-hand endpoints and sometimes at the left. This makes it hard to compute Upper(n) (or Lower(n)).

However, by Theorem 1.3.2, we can use any convenient set of evaluation points for our Riemann sum. As noted earlier, right-hand endpoints x_k are convenient because the general formula is fairly simple. In this case

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

and so

$$x_k = a + k\Delta x = 1 + \frac{2k}{n}.$$

So

$$\begin{aligned} f(x_k) = f\left(1 + \frac{2k}{n}\right) &= -\left(1 + \frac{2k}{n}\right)^2 + 4\left(1 + \frac{2k}{n}\right) - 3 \\ &= -\left(1 + \frac{4k}{n} + \frac{4k^2}{n}\right) + \left(+\frac{8k}{n}\right) - 3 \\ &= \frac{4k}{n} - \frac{4k^2}{n^2}. \end{aligned}$$

The general form of the **right-hand Riemann sum** is:

$$\boxed{\text{Right}(n) = \sum_{k=1}^{n} f(x_k)\Delta x.}$$

In our case, using our work above

$$\begin{aligned} \text{Right}(n) = \sum_{k=1}^{n} f(x_k)\Delta x &= \sum_{k=1}^{n} \left(\frac{4k}{n} - \frac{4k^2}{n^2}\right)\frac{2}{n} \\ &= \frac{8}{n^2}\sum_{k=1}^{n} i - \frac{8}{n^3}\sum_{k=1}^{n} k^2 \\ &= \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \left(4 + \frac{4}{n}\right) - \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) \\ &= \frac{4}{3} - \frac{4}{3n^2}. \end{aligned}$$

By Definition 1.3.3 we have

$$\text{Area} = \lim_{n\to\infty} \text{Right}(n) = \lim_{n\to\infty} \left(\frac{4}{3} - \frac{4}{3n^2}\right) = \frac{4}{3}.$$

Cool!

YOU TRY IT 1.3 (Notation Practice). Fill in the following table for the Riemann sums using regular partitions and right-hand endpoints. *Do not try to evaluate the sums.*

$f(x)$	$[a,b]$	Δx	$x_k = a + k\Delta x$	$f(x_k)$	$\text{Right}(n) = \sum_{k=1}^{n} f(x_k)\Delta x$
$x^2 - 1$	$[0,2]$				
$2(x-1)^2$	$[1,4]$				
$\sin(x)$	$[0,\pi]$				

For $f(x) = 2(x-1)^2$ on $[1,4]$, use algebra to simplify $\text{Right}(n) = \sum_{k=1}^{n} f(x_k)\Delta x$.

Then calculate the area under $f(x) = 2(x-1)^2$ on $[1,4]$ by evaluating $\lim_{n\to\infty} \text{Right}(n)$.

ANSWER TO YOU TRY IT 1.3. Area = 18.

Left-hand Riemann Sums. We have been working with right-hand Riemann sums. But we could use left-hand endpoint sums instead. The the kth subinterval is $[x_{k-1}, x_k]$, so its left-hand endpoint is $x_{k-1} = a + (i-1)\Delta x$. The form of a general **left-hand Riemann sum** is

$$\boxed{\text{Left}(n) = \sum_{k=1}^{n} f(x_{k-1})\Delta x.}$$

Because the expression for the left-hand endpoint $x_{k-1} = a + (i-1)\Delta x$ is more awkward to substitute into a function, we will generally use right-hand endpoint sums. However, notice that by adjusting the starting and ending indices of the sum, we can make left-hand sums as simple as right:

$$\text{Left}(n) = \sum_{k=1}^{n} f(x_{k-1})\Delta x = \sum_{k=0}^{n-1} f(x_k)\Delta x. \tag{1.7}$$

YOU TRY IT 1.4. Let $f(x) = 2(x-1)^2$. Determine the expression for $\text{Left}(n)$ by adjusting the indices using (1.7) above. Simplify the expression. Then calculate the area under $f(x) = 2(x-1)^2$ by evaluating $\lim_{n\to\infty} \text{Left}(n)$. and verify that you get the same area as for $\text{Right}(n)$ in You Try It 1.3

ANSWER TO YOU TRY IT 1.4.

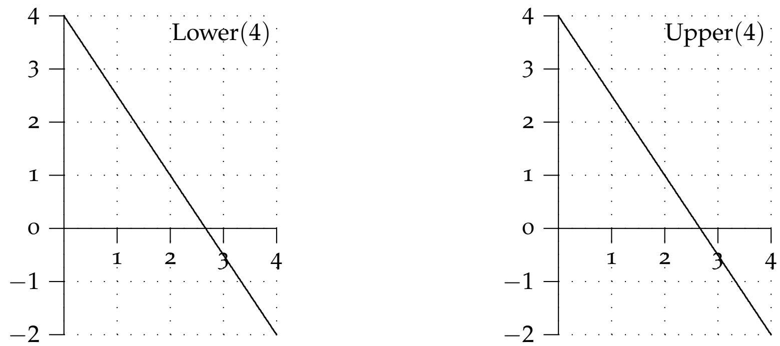
$$\text{Left}(n) = \sum_{k=0}^{n-1} f(x_k)\Delta x = \sum_{k=0}^{n-1} 2\left(\frac{3k}{n}\right)^2 \frac{3}{n} = \frac{54}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right) = 18 - \frac{27}{n} + \frac{9}{n^2}.$$

So the area is $\lim_{n\to\infty} \text{Left}(n) = 18$.

YOU TRY IT 1.5. This problem asks you to extend what we have been doing above.

Functions with negative values. There's no reason why in a Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x$ the function $f(x)$ needs to be non-negative.

(*a*) Using the two graphs of f below, draw $\text{Lower}(4)$ (the lower Riemann sum) and $\text{Upper}(4)$ (upper sum) and evaluate each. Note: some heights and 'areas' will be negative!



(*b*) Using the graph estimate Lower(4) and Upper(4). No summation formulæ are needed.

(*c*) What do these sums represent geometrically?

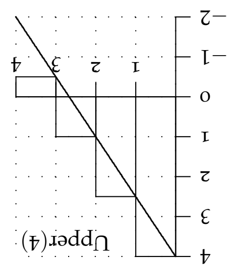
(*d*) The function $f(x)$ is a straight line in this problem. Figure out the equation of $f(x)$.

(*e*) Why does the sum Lower(n) use right endpoints?

(*f*) Set up and simplify Lower(n) thinking of it as Right(n).

(*g*) Evaluate $\lim_{n\to\infty} \text{Lower}(n)$. How is your answer related to the two triangles in the original graph?

ANSWER TO YOU TRY IT 1.5. (*a*) For Upper(4), notice that the bases of the rectangles are always on the x axis, which may not be the bottom of the grid!!!



(*b*) $U(4) = 4 \cdot 1 + 2.5 \cdot 1 + 1 \cdot 1 + (-0.5) \cdot 1 = 7.0$.

(*c*) The rectangles below the axis produce 'negative' area, so the result is **net area**, that is area above the x-axis minus the area below it.

(*d*) $f(x) = -\frac{3}{2}x + 4$.

(*e*) $f(x)$ is decreasing so the lowest point in each interval is at the right end.

(*f*) $\Delta x = \frac{4}{n}$. $x_k = \frac{4i}{n}$. Simplifying we get Lower$(n) = \sum_{k=1}^{n} \left(-\frac{3}{2} \cdot \frac{4i}{n} + 4\right) \frac{4}{n} = 4 - \frac{12}{n}$.

(*g*) $\lim_{n\to\infty} \text{Lower}(n) = 4$.

YOU TRY IT 1.6 (Putting it all together). Suppose that $y = 2x + 2$ on the interval $[-2, 1]$.

(*a*) What is the formula for the right-hand Riemann sum Right(n)?

(*b*) Find $\lim_{n\to\infty} \text{Right}(n)$.

ANSWER TO YOU TRY IT 1.6. (*a*) Right$(n) = \sum_{k=1}^{n} \left[\frac{6i}{n} - 2\right] \cdot \frac{3}{n}$. (*b*) $\lim_{n\to\infty} \text{Right}(n) = 3$.

YOU TRY IT 1.7. Find the general formula for a regular right-hand Riemann sums Right(n) for the following functions and intervals. Make sure to simplify $f(x_k)$. Then use the summation formulæ to simplify Right(n) as much as possible.

(*a*) $f(x) = 4x^3 - 5$ on $[0, 2]$.

(*b*) $f(x) = x^2 - x$ on $[1, 4]$.

(*c*) Evaluate Right(100) for both Riemann sums above. Determine $\lim_{n\to\infty} \text{Right}(n)$ for each.

ANSWER TO YOU TRY IT 1.7. (*a*) $\sum_{k=1}^{n}\left[4\left(\frac{2i}{n}\right)^3-5\right]\cdot\frac{2}{n} = 6+\frac{32}{n}+\frac{16}{n^2}$. (*b*) $\sum_{k=1}^{n}\left[\frac{9k^2}{n^2}+\frac{3i}{n}\right]\cdot\frac{3}{n} = \frac{27}{2}+\frac{18}{n}+\frac{9}{2n^2}$. (*c*) For (a) $\text{Right}(100) = 6+0.32+0.0016 = 6.3216$. For (b) $\text{Right}(100) = 13.5+0.18+0.00045 = 13.68045$. Take the limits: (a) $\lim_{n\to\infty}\text{Right}(n) = 6$; (b) $\lim_{n\to\infty}\text{Right}(n) = 13.5$.

WEBWORK: Click to try Problems 18 through 19. Use GUEST login, if not in my course.

1.4 *The Definite Integral*

Everything has worked out nicely, especially for right-hand Riemann sums using regular partitions. But we started with very general Riemann sums of the form $\sum_{k=1}^{n} f(c_k)\Delta x_k$. The next definition reverts back to this general setup.

DEFINITION 1.4.1. Let f be a function defined at each point in the closed interval $[a,b]$. Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let $\Delta x_k = x_k - x_{k-1}$. Let c_k be any sample point in the interval $[x_{k-1}, x_k]$. If

$$\lim_{\text{all } \Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists, then we say that f is **integrable** on $[a,b]$. If the limit exists, it is denoted by

$$\lim_{\text{all } \Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx,$$

where a and b are the **lower** and **upper limits of integration**. We say that the limit of the Riemann sums, if it exists, is the **definite integral of** f **from** a **to** b.

This definition should remind you of **in**definite integrals. There is a connection that we will see a bit later.

Now in light of Theorem 1.3.2 about upper and lower sums of continuous functions, the following result is not too surprising.

THEOREM 1.4.2 (Continuity implies Integrability). If f is continuous on $[a,b]$, then f is integrable.

The proof of this result is hard. Take Math 331! But Theorem 1.3.2 provides the intuition. Theorem 1.3.2 says that for continuous functions $\lim_{n\to\infty}\text{Upper}(n) = \lim_{n\to\infty}\text{Lower}(n)$ the upper and lower sums same value. Now these are sums using *regular* partitions and as $n \to \infty$, we have that $\Delta x \to 0$. However, Definition 1.4.1 applies to any and *all* partitions. So the new theorem, Theorem 1.4.2, is a much more general result than Theorem 1.3.2 and there is some work to do in proving it.

Take-home Message. Theorem 1.4.2 says that if f is continuous, the limit for any sequence of Riemann sums exists and we get the same number as long as the widths Δx_k of all subintervals go to 0. So to actually compute $\int_a^b f(x)\,dx$ we might as well choose the *most convenient* partitions and sample points. Typically these are right-hand Riemann sums where the partition is regular, so $\Delta x = \frac{b-a}{n}$ and $c_k = x_k$.

EXAMPLE 1.4.3. Determine $\int_{-4}^{2} \frac{x}{2}\,dx$.

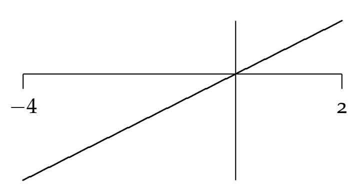


Figure 1.19: The graph of $f(x) = \frac{1}{2}x$ on $[-2,4]$.

Solution. $f(x) = \frac{1}{2}x$ is continuous (it is polynomial, in fact linear), but it is not non-negative. By Theorem 1.4.2 we know that f is integrable. To find the value of the integral, we use a convenient Riemann sum, Right(n).

For n subintervals,

$$\begin{aligned} \Delta x &= \frac{b-a}{n} = \frac{2-(-4)}{n} = \frac{6}{n} \\ x_k &= a + k\Delta x = -4 + \frac{6k}{n} \\ f(x_k) &= \frac{x_k}{2} = -2 + \frac{3k}{n} \end{aligned}$$

Look carefully at the drawing in Figure 1.20. The Riemann sum rectangles ALWAYS START on the x-axis and go up or down to the graph of f. They do not start at the bottom of the picture. Notice that one of the rectangles happens to have a height of 0.

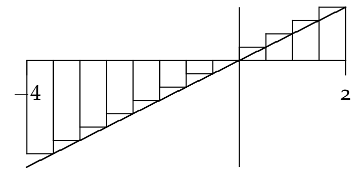


Figure 1.20: Right(n) for $f(x) = \frac{1}{2}x$ on $[-2, 4]$.

$$\begin{aligned} \text{Right}(n) = \sum_{k=1}^{n} f(x_k)\Delta x &= \sum_{k=1}^{n} f\left(-4 + \frac{6k}{n}\right) \cdot \frac{6}{n} \\ &= \frac{6}{n}\sum_{k=1}^{n}\left(-2 + \frac{3k}{n}\right) \\ &= \frac{6}{n}\sum_{k=1}^{n} -2 + \frac{6}{n} \cdot \frac{3}{n}\sum_{k=1}^{n} i \\ &= \frac{6}{n}(-2n) + \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) \\ &= -12 + 9 + \frac{9}{n} \\ &= -3 + \frac{9}{n}. \end{aligned}$$

So

$$\int_{-4}^{2} \frac{x}{2}\,dx = \lim_{n\to\infty} \text{Right}(n) = \lim_{n\to\infty} -3 + \frac{9}{n} = -3.$$

Notice that the answer to Example 1.4.3 is negative so it cannot represent an ordinary area which must be non-negative. The integral represents the 'net area:' the area above the x-axis minus the area below the axis. In this case, the areas above and below the x-axis are two triangles and we see that the difference in their areas is

$$\text{area above the axis} - \text{area below the axis} = \tfrac{1}{2}(2)(1) - \tfrac{1}{2}(4)(2) = -3$$

which checks with the integral. Now we did not need Riemann sums to compute net areas of triangles. But as soon as we have curved regions, integrals (using Riemann sums) are the only method we have of computing such net areas.

Finally, when f is non-negative, the integral is equal to the area in the traditional sense. In other words, we have solved the area problem.

DEFINITION 1.4.4 (Area as an Integral). If f is continuous and non-negative on the closed interval $[a, b]$, then the area, then the area bounded above by $f(x)$, below by the x-axis, and by the vertical lines $x = a$ and $x = b$ is

$$\text{Area}(\text{under } f) = \int_a^b f(x)\,dx.$$

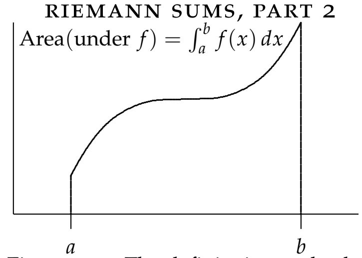


Figure 1.21: The definite integral solves the area problem.

YOU TRY IT 1.8. Show that the area under the parabola $y = f(x) = 1 - x^2$ on the interval $[-1, 1]$ is $\frac{4}{3}$ using

$$\int_{-1}^{1} 1 - x^2\,dx = \lim_{n\to\infty} \text{Right}(n).$$

WEBWORK: Click to try Problems 20 through 27. Use GUEST login, if not in my course.