1.5 *Two Methods to Evaluate Definite Integrals*

At the moment we have two methods to evaluate $\int_a^b f(x)\,dx$.

1. Definition 1.4.1 says that we we can evaluate $\int_a^b f(x)\,dx$ as a limit, most conveniently as $\lim_{n\to\infty} \text{Right}(n)$. This is what you did in YOU TRY IT 1.8 and what we have carried out in other problems.

2. On the other hand, we can use Definition 1.4.4 and ask whether $\int_a^b f(x)\,dx$ represents some well-known area. If so, we can use the corresponding area formula to evaluate the integral. Below you will find several such examples.

EXAMPLE 1.5.1. Determine $\displaystyle\int_3^7 2\,dx$. (See Figure 1.22.)

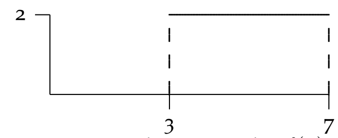

Figure 1.22: The area under $f(x) = 2$ is a rectangle.

Solution. The area under the graph of the constant function $f(x) = 2$ is a rectangle. So

$$\int_3^7 2\,dx = \text{Area(rectangle)} = b \times h = 4 \times 2 = 8.$$

EXAMPLE 1.5.2. Determine $\displaystyle\int_1^5 2x+1\,dx$. (See Figure 1.23.)

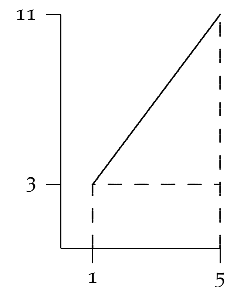

Figure 1.23: The area under $f(x) = 2x+1$ is a trapezoid.

Solution. The area under the linear function $f(x) = 2x+1$ is a trapezoid (which can be split into a rectangle and triangle if you have forgotten the area formula for a trapezoid).

$$\int_1^5 2x+1\,dx = \text{Area(trapezoid)} = \tfrac{1}{2}(b_1+b_2) \times h = \tfrac{1}{2}(3+11) \times 4 = 28.$$

Or

$$\int_1^5 2x+1\,dx = \text{Area(triangle + rectangle)} = \tfrac{1}{2}(4)(8) + (4)(3) = 28.$$

EXAMPLE 1.5.3. Determine $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$.

Solution. The area under $f(x) = \sqrt{9-x^2}$ is a quarter-circle of radius 3. So

$$\int_0^3 \sqrt{9-x^2}\,dx = \text{Area(quarter-circle)} = \tfrac{1}{4}(\pi r^2) = \tfrac{9\pi}{4}.$$

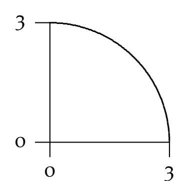

Figure 1.24: The area under $f(x) = \sqrt{9-x^2}$ is a quarter-circle.

EXAMPLE 1.5.4. Determine $\displaystyle\int_0^{2\pi} \sin x\,dx$.

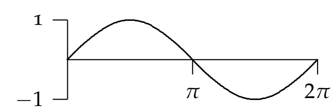

Figure 1.25: The net area under $f(x) = \sin x$ is 0.

Solution. The area above the x-axis in Figure 1.25 is the same as the area below the axis. Thus, the net area is 0 which means

$$\int_0^{2\pi} \sin x\,dx = 0.$$

EXAMPLE 1.5.5. Change the interval in the previous problem: Determine $\displaystyle\int_0^{\pi} \sin x\,dx$.

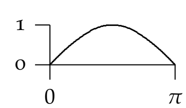

Figure 1.26: We don't yet know how to find the area under $f(x) = \sin x$ on $[0, \pi]$.

Solution. Now we cannot take advantage of symmetry and we do not have an 'area formula' for the area under the sine function. We would need to use Riemann sums. You can check that

$$\text{Right}(n) = \sum_{k=1}^{n} \sin\left(\frac{k\pi}{n}\right) \cdot \frac{\pi}{n}.$$

However, we don't have a 'summation formula' to simplify this sum. For the time being, we are stuck!

EXAMPLE 1.5.6. Determine $\int_{-1}^{1} -\sqrt{1-x^2}\,dx$.

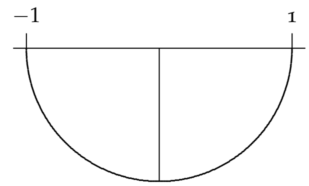

Figure 1.27: The area between $f(x) = -\sqrt{1-x^2}$ and the x-axis is a semi-circle below the axis.

Solution. The area under $f(x) = -\sqrt{1-x^2}$ is a unit semi-circle *below* the x-axis. So

$$\int_{-1}^{1} -\sqrt{1-x^2}\,dx = -\text{Area(semi-circle)} = -\tfrac{1}{2}(\pi r^2) = -\tfrac{\pi}{2}.$$

Take-home Message. Example 1.5.5 points out that even though the definite integral 'solves' the area problem, we must still be able to evaluate the Riemann sums involved. If the region is not a familiar one and we can't determine

$$\lim_{\text{all } \Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

then we are stuck in trying to evaluate $\int_a^b f(x)\,dx$. In other words, we must find yet another method to evaluate definite integrals.

1.6 Useful Properties of the Definite Integrals

The definition of the definite integral of f on $[a, b]$ requires that $a < b$. However, it is convenient to extend this definition to the two other cases: $a = b$ and $a > b$. In the first of these cases, when $a = b$ the geometry tells us that the area should be 0. (See Figure 1.28.) When $a > b$ we can think of the Riemann sums taking place in reverse going from right to left: $\Delta x = \frac{b-a}{n}$ is now *negative* since $a > b$ so the terms in the Riemann sum all change sign, so the integral changes sign when the limits are reversed. These two observations are summarized in the following definition.

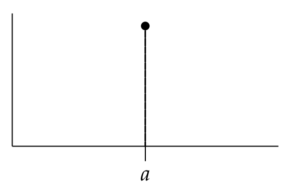

Figure 1.28: The area below a single point is 0. Equivalently, $\int_a^a f(x)\,dx = 0$.

DEFINITION 1.6.1. We extend the definition of the definite integral as follows:

1. If f is defined at a, then $\boxed{\int_a^a f(x)\,dx = 0.}$
2. If f is integrable on $[a, b]$, then $\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.}$

EXAMPLE 1.6.2. In Example 1.4.3, we saw that $\int_{-1}^{2} \frac{x}{2}\,dx = -3$ so reversing the limits and using part 2 of Definition 1.6.1 we obtain $\int_{2}^{-1} \frac{x}{2}\,dx = 3$. Using part 1 of Definition 1.6.1 we find $\int_1^1 e^x\,dx = 0$.

We can generalize the result in Example 1.5.1. Observe that if $f(x) = k$ is a constant function, then the area it determines is a rectangle of height k and width $b - a$ (see Figure 1.29), so

$$\boxed{\int_a^b k\,dx = k(b-a).}$$

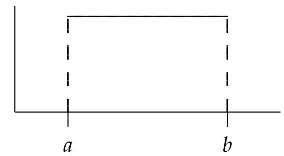

Figure 1.29: The area formed by a constant function is a rectangle.

Since we have assumed that we can compute areas of non-overlapping regions by summing the areas of the individual pieces (see Basic Area Property 3 on page 1) as in Figure 1.30. Consequently,

THEOREM 1.6.3 (Additivity). If f is integrable on the three closed intervals determined by a, b, and c, then

$$\boxed{\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.}$$

Note: The order of the points does not matter.

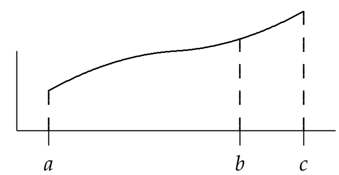

Figure 1.30: If f is integrable on $[a,b]$ and $[b,c]$, then f is integrable on $[a,c]$.

EXAMPLE 1.6.4. Suppose that $\int_0^{12} f(x)\,dx = 25$ and $\int_0^8 f(x)\,dx = 14$, determine $\int_8^{12} f(x)\,dx$.

Solution. From Theorem 1.6.3 we have $\int_0^8 f(x)\,dx + \int_8^{12} f(x)\,dx = \int_0^{12} f(x)\,dx$. Substituting the given values we get

$$14 + \int_8^{12} f(x)\,dx = 25 \quad \text{so, it follows that} \quad \int_8^{12} f(x)\,dx = 11.$$

EXAMPLE 1.6.5. Determine $\int_{-2}^{1} |x|\,dx$.

Solution. This time we use a bit of geometry. From Figure 1.31 we see that we can divide the the region between the graph and the function into two triangles. So using Theorem 1.6.3 we have

$$\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_0^1 |x|\,dx = \tfrac{1}{2}(2)(2) + \tfrac{1}{2}(1)(1) = 2.5$$

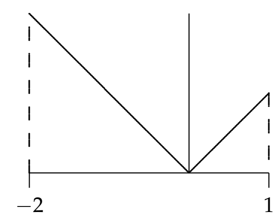

Figure 1.31: To determine $\int_{-2}^{1} |x|\,dx$, divide the the region between the graph and the function into two triangles.

Because a definite integral is the limit of a (Riemann) sum, it has both the distributive and associative properties:

$$\sum_{k=1}^{n} kf(c_k)\Delta x_k = k\sum_{k=1}^{n} f(c_k)\Delta x_k$$

and

$$\sum_{k=1}^{n} [f(c_k) + g(c_k)]\Delta x_k = \sum_{k=1}^{n} f(c_k)\Delta x_k + \sum_{k=1}^{n} g(c_k)\Delta x_k.$$

Taking limits, we see

THEOREM 1.6.6 (Linearity). If f and g are integrable on $[a,b]$ and k is any constant, then $kf(x)$ and $f(x) \pm g(x)$ are integrable on $[a,b]$. Further,

1. $$\boxed{\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.}$$

2. $$\boxed{\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.}$$

EXAMPLE 1.6.7. Suppose that $\int_0^{12} f(x)\,dx = 25$ and $\int_0^{12} g(x) = -4$. Evaluate each of the following definite integrals.

(a) $\int_0^{12} 3f(x)\,dx$ (b) $\int_0^{12} 2f(x) - 9g(x)\,dx$ (c) $\int_{12}^{0} 3 - g(x)\,dx$ (d) $\int_{12}^{12} x^2 f(x)\,dx$

Solution. Use Theorem 1.6.6 for part (a).

(a) $\int_0^{12} 3f(x)\,dx = 3\int_0^{12} f(x)\,dx = 3(25) = 75.$

(b) Use both parts of Theorem 1.6.6

$$\int_0^{12} 2f(x) - 9g(x)\,dx = \int_0^{12} 2f(x)\,dx - \int_0^{12} 9g(x)\,dx$$
$$= 2\int_0^{12} f(x)\,dx - 9\int_0^{12} g(x)\,dx = 2(25) - 9(-4) = 86.$$

(c) This time use both Definition 1.6.1 and Theorem 1.6.6 as well as our observation about constant functions.

$$\int_{12}^{0} 3 - g(x)\,dx = -\int_0^{12} 3 - g(x)\,dx$$
$$= -\int_0^{12} 3\,dx - \int_0^{12} g(x)\,dx = (12)(3) - 4 = 32.$$

(d) This time notice that both endpoints are equal, so $\int_{12}^{12} x^2 f(x)\,dx = 0$.

Another important property of definite integrals is that they preserve inequalities.

THEOREM 1.6.8. Assume that f and g are both integrable on $[a, b]$.

1. If $f(x) \geq 0$ for all x in $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx \geq 0.}$$

2. More generally, if $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.}$$

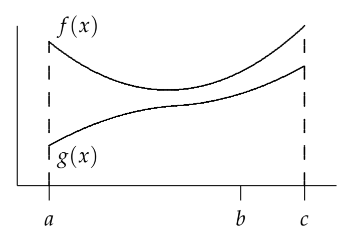

Figure 1.32: If $f(x) \geq g(x)$ on $[a, b]$ and both are integrable, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

While we won't give proofs of either part of Theorem 1.6.8, the intuition is clear. For part 1, Since $f(x) \geq 0$ on the interval, then for any point c_k in $[a, b]$, we have $f(c_k) \geq 0$. So

$$\sum_{k=1}^{n} f(c_k)\Delta x_k \geq \sum_{k=1}^{n} 0\Delta x_k = 0.$$

Taking limits gives $\int_a^b f(x)\,dx \geq 0$. For part 2, since $f(x) \geq g(x)$, then $f(x) - g(x) \geq 0$ for all x in $[a, b]$. So by part 1,

$$\int_a^b f(x) - g(x)\,dx \geq 0$$

and so by the linearity theorem (Theorem 1.6.6) we have

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

YOU TRY IT 1.9. Determine $\int_{-5}^{4} f(x)\,dx$ for the function f in Figure 1.32.

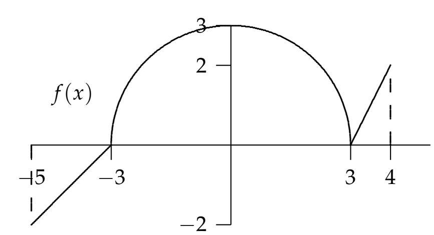

Figure 1.33: Find the area between f and the x-axis.

ANSWER TO YOU TRY IT 1.9. $\frac{9\pi}{2} - 1$.

WEBWORK: Click to try Problems 40 through 43. Use GUEST login, if not in my course.

The Fundamental Theorems of Calculus

2.1 The Fundamental Theorem of Calculus, Part II

Recall the *Take-home Message* we mentioned earlier. Example 1.5.5 points out that even though the definite integral 'solves' the area problem, we must still be able to evaluate the Riemann sums involved. If the region is not a familiar one and we can't determine

$$\lim_{\text{all } \Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

then we are stuck in trying to evaluate $\int_a^b f(x)\,dx$. In other words, *we must find another method to evaluate definite integrals.* We now make the connection between antiderivatives and definite integrals. To do this, we will need to use the Mean Value Theorem in the following form:

THEOREM 2.1.1 (MVT: The Mean Value Theorem)**.** Assume that

1. F is continuous on the closed interval $[x_{k-1}, x_k]$;
2. F is differentiable on the open interval (x_{k-1}, x_k);

Then there is some point c_k between x_{k-1} and x_k so that

$$F'(c_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}.$$

This is equivalent to saying $F(x_k) - F(x_{k-1}) = F'(c_k) \cdot (x_k - x_{k-1})$. Or using the notation of Riemann sums,

$$F(x_k) - F(x_{k-1}) = F'(c_k)\Delta x_k.$$

THEOREM 2.1.2 (FTC Part II). Assume that f is continuous on $[a, b]$ and that F is an antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Notes: We will cover what your text calls Part I of the FTC shortly.
Also, recall: F is an antiderivative of f means that $F' = f$ on $[a, b]$.

Before we do the proof, let's look at an example so you can appreciate what this theorem says.

EXAMPLE 2.1.3. Let $f(x) = x^2$ on $[0, 2]$. An antiderivative is $F(x) = \frac{1}{3}x^3$. So Theorem 2.1.2 says

$$\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}.$$

Wow! That's a heck of a lot simpler than doing a limit of Riemann sums. Now to 'pay' for this convenience, we need to spend a few minutes working through the proof of the theorem. But it will pay big dividends.