Sequences

Our ultimate goal by the end of the course is to approximate functions by using polynomials with an infinite number of terms. Such approximations are what allows your calculator to evaluate log, trig, and exponential functions (which are types of *transcendental functions*[1]). Similar approximations are used in data compression so that your iPod can play music files that are relatively small. We want to use polynomials rather than trig functions, logs, or exponentials because they are easier to work with. The down side is that this forces us to deal with infinity, or more precisely, limits of "infinite sums which turn out to be very interesting!

[1] Roughly speaking, there are two types of functions: *algebraic* and *transcendental*. The algebraic functions are created by using the elementary operations of addition, subtraction, multiplication, division, and root extraction. Transcendental functions are all those functions that are *not* algebraic. In other words, a function which "transcends," i.e., cannot be expressed in terms of, algebra.

Introduction

Our first topic is sequences. You have probably seen sequences on SAT or IQ tests where you had to figure out the next term or the general pattern. Here are several examples.

EXAMPLE 12.0.1. See if you can figure out the next term and the general formula for the nth term for each sequence or list of numbers.

a. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

b. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

c. $0, -1, 4, -9, 16, \ldots$

d. $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \ldots$

e. $1, 2, 6, 24, 120, \ldots$

DEFINITION 12.0.2. A **sequence** of real numbers is a function $f(n)$ whose domain is the set of all positive integers n. Notation: Instead of using $f(n)$ we usually use a_n and indicate the entire sequence by $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$. More generally, a sequence can start with any integer m in which case the domain consists of all integers $n \geq m$. ($m = 0$ is a common starting value.)

Sequences can be described in two ways. Often we will use an **explicit formula** like we would with an ordinary function). For instance

EXAMPLE 12.0.3 (Continued). Here are the functions (formulæ) describing the sequences in Example 12.0.1.

a. $f(n) = \frac{1}{2^n}, n \geq 0; \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$.

b. $a_n = \frac{1}{n}, n \geq 1; \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

c. $\{(-1)^n n^2\}_{n=0}^{\infty}$.

d. $\left\{1 + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

e. $\{n!\}_{n=1}^{\infty}$.

EXAMPLE 12.0.4. The sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ can be described in several ways. For example, we might use $a_n = f(n) = \frac{n-1}{n}$ for $n \geq 1$, so the sequence would be $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$. Here we can compute a_n explicitly from the given formula.

Such a formula is not unique. We might start with $n = 0$ and use $b_n = g(n) = \frac{n}{n+1}$ for $n \geq 0$, so the sequence would be $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$.

Another way that sequences are defined is with a **recurrence relation**. We specify the first term of the sequence and give a general rule for computing the next term of the sequence from the previous ones. For example,

$$a_1 = 1,\ a_{n+1} = a_n^2 - 1.$$

The first few terms are

$$1, 0, -1, 0, -1, 0, -1, \ldots$$

Another such sequences is the **factorial** sequence (function). . The function n-factorial is denoted by $n!$. We define

$$a_0 = 0! = 1, \; a_{n+1} = (n+1)! = n! \cdot (n+1)$$

So the first few terms are

$$\begin{aligned}
0! &= 1 \\
1! &= 0! \cdot 1 = 1 \cdot 1 = 1 \\
2! &= 1! \cdot 2 = 1 \cdot 2 = 2 \\
3! &= 2! \cdot 3 = 1 \cdot 2 \cdot 3 = 6 \\
4! &= 3! \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24
\end{aligned}$$

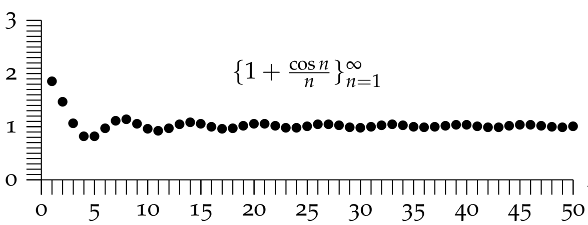
You can see that we can also give an explicit formula for $n!$:

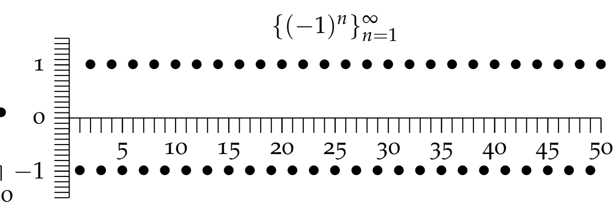
$$n! = 1 \cdot 2 \cdots n$$

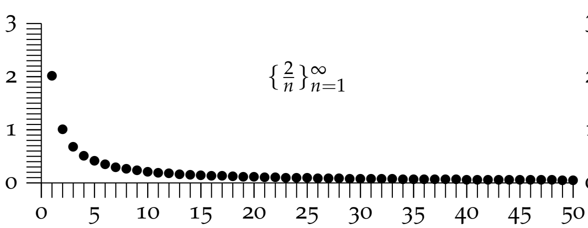
For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

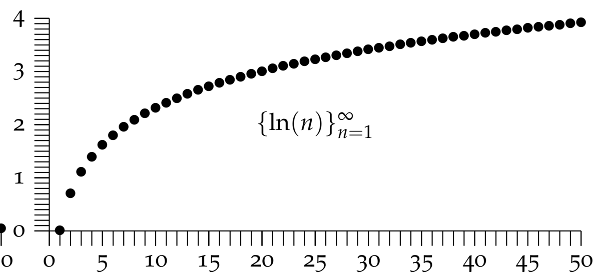
YOU TRY IT 12.1. The factorial function gets large very fast. Try some values on your calculator. What is the smallest value of n so that $n! > 1,000,000$?

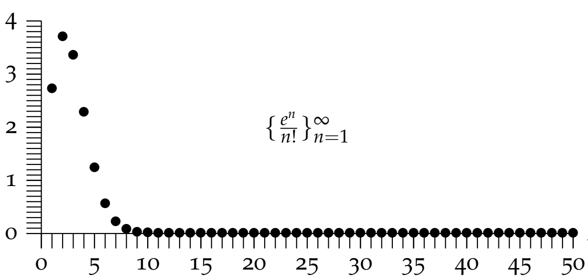
Though we don't often do it, we can graph sequences. Here are some examples.

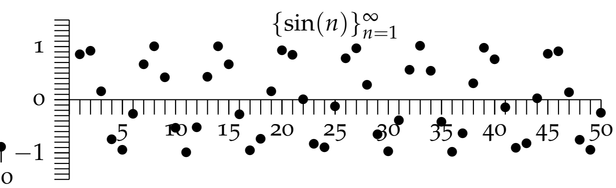


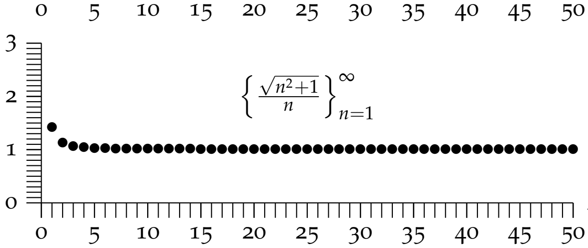


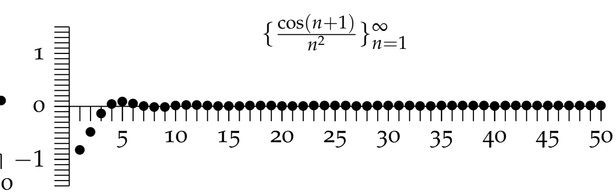


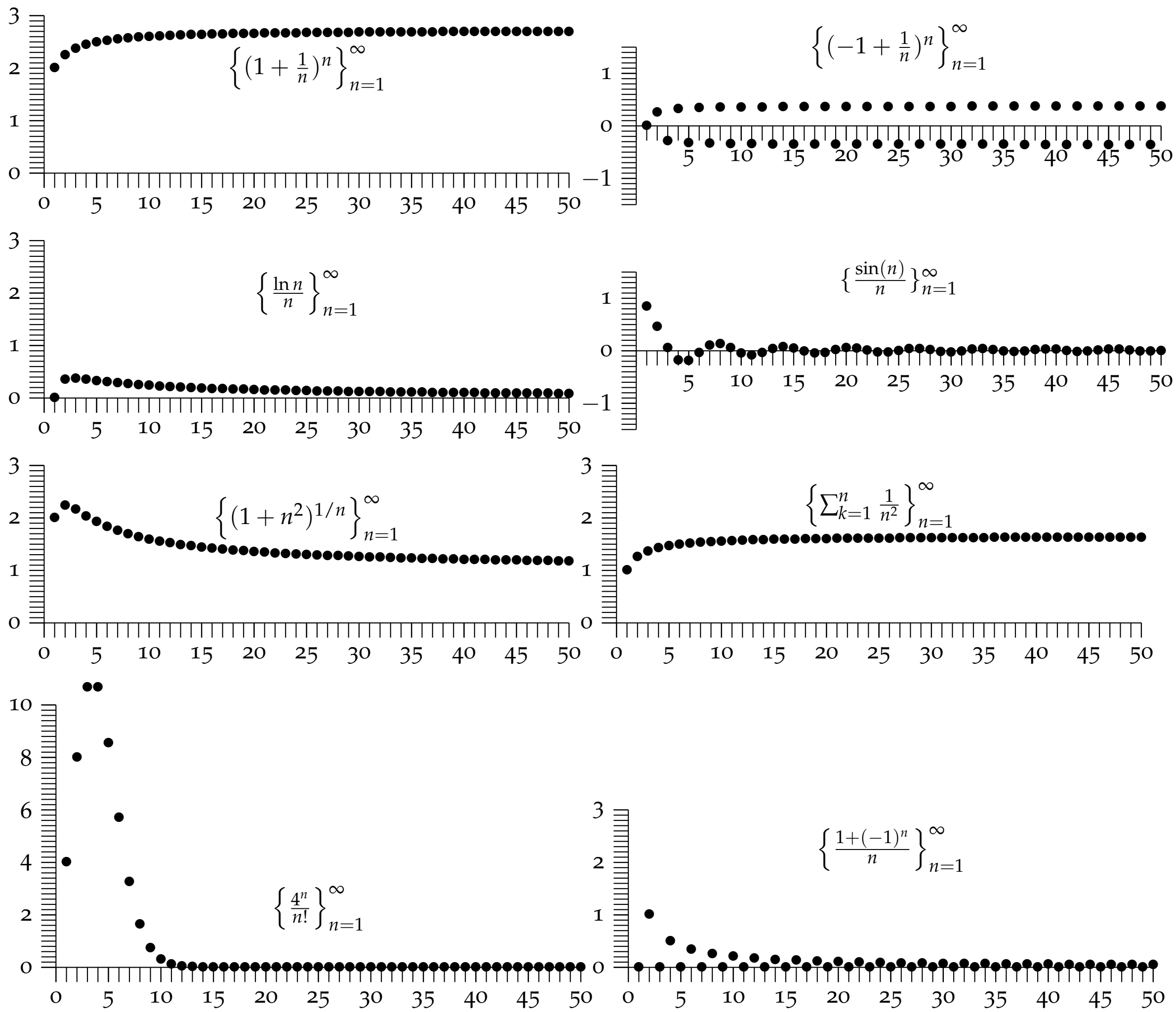












Caution: One important thing to notice: These sequences (functions) are *not continuous* because they are not defined on intervals. Their graphs are just an infinite series of dots, one for each integer in the domain of the sequence.

Limits of Sequences

In some of the graphs above, as n gets large, the terms in the sequence seem to be approaching a particular value. For example the next to last sequence $\left\{\frac{4^n}{n!}\right\}_{n=1}^{\infty}$ appears to approach 0 as n gets large. On the other hand, $\{\sin n\}_{n=1}^{\infty}$ does not appear to approach any particular value as n gets large. We can adapt the language of limits to this situation.

DEFINITION 12.0.5 (Informal). A sequence $\{a_n\}_{n=1}^{\infty}$ has a **limit** L if we can make a_n arbitrarily close to L by taking n sufficiently large. We denote this by writing

$$\lim_{n\to\infty} a_n = L.$$

EXAMPLE 12.0.6. Here's a familiar sequence from 'back in the day' when we were working with Riemann sums.

(*a*) Let

$$\{u_n\}_{n=1}^{\infty} = \left\{\frac{n(n+1)}{2n^2}\right\}_{n=1}^{\infty}.$$

We evaluate this limit easily by using a bit of algebra (or use l'Hôpital's rule):

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

Another way to solve algebraic limits at infinity like this one is to focus on the highest powers in algebraic or rational functions.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n^2+n}{n^2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^2}{2n^2} = \frac{1}{2}.$$

Still another way is to use l'Hôpital's rule, but see Theorem 12.0.7.

(*b*) Another similar example we saw was

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{n(n+1)(2n+1)}{6n^3} \right\}_{n=1}^{\infty}.$$

Since this is an algebraic limit at infinity, let's use highest powers:

$$\begin{aligned} \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^2}{6n^2} = \frac{1}{3}. \end{aligned}$$

(*c*) Here's a 'backwards' sequence as an illustration of how general the sequence concept is: The index goes backwards to $-\infty$. Let

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{\sqrt{5n^2+n+1}}{2n+3} \right\}_{n=-1}^{-\infty}.$$

Since this is a limit at infinity, we can focus on the 'highest powers' to simplify the limit:

$$\lim_{n\to-\infty} a_n = \lim_{n\to-\infty} \frac{\sqrt{5n^2+n+1}}{2n+3} \overset{\text{HPwrs}}{=} \lim_{n\to-\infty} \frac{\sqrt{5n^2}}{2n}.$$

Now here is where you need to be very careful: Remember that $\sqrt{x^2} = |x|$, NOT x. (Try a few negative values of x to see why.) Further, when x is negative, $|x| = -x$. So continuing the calculation above, since the n-values are negative

$$\lim_{n\to-\infty} a_n = \lim_{n\to-\infty} \frac{\sqrt{5n^2}}{2n} = \lim_{n\to-\infty} \frac{|n|\sqrt{5}}{2n} = \lim_{n\to-\infty} \frac{-n\sqrt{5}}{2n} = -\frac{\sqrt{5}}{2}.$$

YOU TRY IT 12.2. Return to the graphs of the sequences given earlier. By inspecting the graphs (no calculations), which appear to have limits and which do not?

The following do not have limits: $\{(-1)^n\}_{n=1}^{\infty}$, $\{\ln n\}_{n=1}^{\infty}$, $\{\sin n\}_{n=1}^{\infty}$, $\left\{\left(-1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$.

Some sequence limits are more challenging. Consider the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$. Does it have a limit? We might try to evaluate

$$\lim_{n\to\infty} \frac{\ln n}{n}$$

by using l'Hôpital's rule. However, to apply l'Hôpital's rule the numerator and denominator of the sequence need to be differentiable. But in a sequence, these functions are not even continuous, let alone differentiable. Fortunately there is a way around this.

THEOREM 12.0.7 (Key Fact). Suppose that f is a function so that $\lim_{x\to\infty} f(x) = L$. If a_n is a sequence such that $f(n) = a_n$ for all integers n in the domain of the sequence, then $\lim_{n\to\infty} a_n = L$, too.

Essentially this says that if we can 'convert' a sequence to a corresponding function of x and evaluate the resulting limit, then the sequence has the same limit as the function.

EXAMPLE 12.0.8. Return to the sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$. The sequence can be described by the formula $a_n = f(n) = \frac{\ln n}{n}$. So if we let $f(x) = \frac{\ln x}{x}$ (for $x > 0$), then we can try to evaluate the limit as $x \to \infty$ using any method from Calculus I.
In particular we can apply l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$$

Therefore by the Key Fact Theorem 12.0.7 above,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0,$$

also!

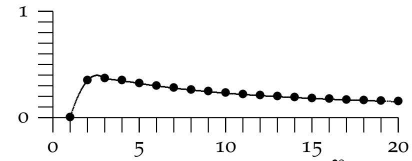


Figure 12.1: The sequence $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ and the function $f(x) = \frac{\ln x}{x}$ that 'connects the dots.' Both appear to have the same limit at infinity.

YOU TRY IT 12.3. Here are several that you should try now. Many of these will look familiar from our recent work with l'Hôpital's rule.

(*a*) $\displaystyle\lim_{n\to\infty} \frac{2n}{n+1}$.

(*b*) $\displaystyle\lim_{n\to\infty} \frac{1-n^2}{6n+7}$.

(*c*) $\displaystyle\lim_{n\to\infty} \frac{e^n}{n^2+1}$.

(*d*) $\displaystyle\lim_{n\to\infty} (1+2n)^{1/n}$.

(*e*) $\displaystyle\lim_{n\to\infty} \frac{\ln\sqrt{n}}{n}$.

(*f*) $\displaystyle\lim_{n\to\infty} (\sqrt{n})^{1/n}$.

(*g*) $\displaystyle\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n/2}$.

(*h*) $\displaystyle\lim_{n\to\infty} (\ln(3n+5) - \ln(4n+6))$.

ANSWER TO YOU TRY IT 12.3. (a) 2; (b) DNE $(-\infty)$; (c) DNE $(+\infty)$; (d) 1; (e) 0; (f) 1; (g) $e^{-1/2}$; (h) $\ln(3/4)$.

WEBWORK: Click to try Problems 132 through 135. Use GUEST login, if not in my course.

Important Limits

The sequence $\left\{\left(1+\frac{k}{n}\right)^n\right\}_{n=1}^{\infty}$

Finding the limit of the sequence $\left\{\left(1+\frac{k}{n}\right)^n\right\}_{n=1}^{\infty}$ should remind you of some earlier work we did with l'Hôpital's rule. Notice that $\lim_{n\to\infty}\left(1+\dfrac{k}{n}\right)^n$ has the indeterminate form 1^{∞}. To determine limits of this form, it is useful to use the natural log function.

Let $y = \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n$. Taking the logs of both sides and switching the order of the limit and the log (the log function is continuous) we get:

$$\ln y = \ln \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = \lim_{n\to\infty}\ln\left(1+\frac{k}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{k}{n}\right).$$

To evaluate the limit we switch to the continuous variable x and put the limit in $\frac{0}{0}$ indeterminate form so that we can eventually use l'Hôpital's rule.

$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{k}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\cdot\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = \frac{k}{1+0} = k.$$

Since $\ln y = k$, then $y = e^k$ which means that $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$.

The sequence $\{n^{1/n}\}_{n=1}^{\infty}$

This time $\lim_{n\to\infty} n^{1/n}$ or $\lim_{n\to\infty}\sqrt[n]{n}$ has the indeterminate form ∞^0. We use the same method as in the previous situation. Let $y = \lim_{n\to\infty} n^{1/n}$. Then

$$\ln y = \ln\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}\ln n^{1/n} = \lim_{n\to\infty}\frac{\ln n}{n}.$$

To evaluate the limit we switch the the continuous variable x and use l'Hôpital's rule since the limit is now in the indeterminate form $\frac{\infty}{\infty}$.

$$\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.$$

Since $\ln y = 0$, then $y = e^0 = 1$ which means that $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}\sqrt[n]{n} = 1$.

The sequence $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$

This sequence is a little simpler to deal with: Notice that

$$0 \le \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots(n-1)\cdot n}{n\cdot n\cdot n\cdots n\cdot n} = \frac{1}{n}\cdot\frac{2\cdot 3\cdots(n-1)\cdot n}{n\cdot n\cdots n\cdot n} \le \frac{1}{n}\cdot 1.$$

This means that

$$0 \le \frac{n!}{n^n} \le \frac{1}{n}.$$

So taking limits and applying the squeeze theorem we obtain

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{n!}{n^n} \le \lim_{n\to\infty}\frac{1}{n} \Rightarrow 0 \le \lim_{n\to\infty}\frac{n!}{n^n} \le 0.$$

Therefore, $\lim_{n\to\infty}\frac{n!}{n^n} = 0$.

YOU TRY IT 12.4. Give an argument that shows $\lim_{n\to\infty}\dfrac{n^n}{n!} = \infty$ (diverges). Hint: Show that $\frac{n^n}{n!} > n$.

Sequences of the form $\{r^n\}_{n=1}^{\infty}$

For each fixed real number r, we can form the sequence $\{r^n\}_{n=1}^{\infty}$. For some values of r, the sequence converges and for others it does not.

EXAMPLE 12.0.9. By inspection we see that

(*a*) $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$ and so $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$.

(*b*) $\left\{\left(-\frac{1}{4}\right)^n\right\}_{n=1}^{\infty} = \left\{-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \dots\right\}$ and so $\lim_{n\to\infty}\left(-\frac{1}{4}\right)^n = 0$.

(*c*) $\{2^n\}_{n=1}^{\infty} = \{2, 4, 8, 16, \dots\}$ and so $\lim_{n\to\infty}(2)^n$ diverges to $+\infty$.

(*d*) $\{(-3)^n\}_{n=1}^{\infty} = \{-3, 9, -27, 81, \dots\}$ and so $\lim_{n\to\infty}(-3)^n$ diverges.

(*e*) $\{1^n\}_{n=1}^{\infty} = \{1, 1, 1, 1, \dots\}$ and so $\lim_{n\to\infty}(1)^n = 1$.

(*f*) $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \dots\}$ and so $\lim_{n\to\infty}(-1)^n$ diverges.

We see that if $|r| < 1$ then the powers of r get small and converge to 0. If $|r| > 1$, then the powers of r get large (without bound) in magnitude and so the sequence diverges. This is summarized below.

Summary of Key Limits

You should know and be able to use all of the following limits.

THEOREM 12.0.10. Summary of important limits.

(*a*) $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$. In particular $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

(*b*) $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}\sqrt[n]{n} = 1$.

(*c*) $\lim_{n\to\infty}\frac{n!}{n^n} = 0$ and $\lim_{n\to\infty}\frac{n^n}{n!} = \infty$ (diverges).

(*d*) Consider the sequence $\{r^n\}_{n=1}^{\infty}$, where r is a real number.

1. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$;
2. If $r = 1$, then $\lim_{n\to\infty} r^n = 1$;
3. Otherwise $\lim_{n\to\infty} r^n$ does not exist (diverges).

EXAMPLE 12.0.11. These Key Limits may be used with some algebraic techniques.

(*a*) $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n/3} = \lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^n\right]^{1/3} = [e^2]^{1/3} = e^{2/3}$.

(*b*) $\lim_{n\to\infty} n^{4/n} = \lim_{n\to\infty}\left[n^{1/n}\right]^4 = [1]^4 = 1$.

(*c*) $\lim_{n\to\infty}\left(\frac{3}{5}\right)^{6n} = \lim_{n\to\infty}\left[\left(\frac{3}{5}\right)^n\right]^6 = [0]^6 = 0$.

(*d*) $\lim_{n\to\infty}(-3)^{2n}\cdot 7^{-n} = \lim_{n\to\infty}\frac{[(-3)^2]^n}{7^n} = \lim_{n\to\infty}\frac{9^n}{7^n} = \lim_{n\to\infty}\left(\frac{9}{7}\right)^n$ Diverges.

A final note about sequences. It should be clear that

$$\text{if } \lim_{n\to\infty} a_n = L \text{ then } \lim_{n\to\infty} a_{n+1} = L \text{ and } \lim_{n\to\infty} a_{n-1} = L \tag{12.1}$$

since the terms in the infinite tails of the sequences are the same.

WEBWORK: Click to try Problems 136 through 137. Use GUEST login, if not in my course.

12.1 *Problems*

1. (*a*) List the first four terms of the sequence $\left\{\frac{n+1}{3n-1}\right\}_{n=1}^{\infty}$.

(*b*) Find a formula for a_n for the sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\}$.

(*c*) Find a formula for a_n for the sequence $\{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \dots\}$.

2. Determine whether the sequence converges or diverges. If it converges, find the limit. Use l'Hôpital's rule where appropriate. Use log properties for (d).

(*a*) $\left\{\dfrac{\sqrt{n}}{1+n}\right\}_{n=1}^{\infty}$ (*b*) $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}_{n=1}^{\infty}$ (*c*) $\left\{\dfrac{\ln n^2}{n}\right\}_{n=1}^{\infty}$ (*d*) $\{\ln(2n+1)-\ln(3n)\}_{n=1}^{\infty}$

3. Find these limit (if they exists):

(*a*) $\{a_n\}_{n=1}^{\infty} = \left\{\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dfrac{3}{n^2}+\cdots+\dfrac{n}{n^2}\right\}_{n=1}^{\infty}$ (*b*) $\{a_n\}_{n=2}^{\infty} = \left\{\displaystyle\int_1^{\infty}\frac{1}{x^n}\,dx\right\}_{n=2}^{\infty}$

4. Find the limits of these sequences. Use the key limits when possible.

(*a*) $\left\{\left(1+\dfrac{3}{n}\right)^n\right\}_{n=1}^{\infty}$ (*b*) $\left\{\ln(2n^2+7)-\ln(5n^2+n)\right\}_{n=1}^{\infty}$ (*c*) $\left\{\dfrac{2\ln(n+1)}{n^2}\right\}_{n=1}^{\infty}$

(*d*) $\left\{\left(\dfrac{2}{3}\right)^n\right\}_{n=1}^{\infty}$ (*e*) $\left\{\left(\dfrac{-3}{2}\right)^n\right\}_{n=1}^{\infty}$ (*f*) $\left\{\dfrac{4n^2-3n+1}{5n^2+7}\right\}_{n=1}^{\infty}$

5. Find the limits of the following sequences.

(*a*) $\left\{\dfrac{3n}{n+1}\right\}_{n=1}^{\infty}$ (*b*) $\left\{\dfrac{1-n^2}{6n+7}\right\}_{n=1}^{\infty}$ (*c*) $\left\{\dfrac{e^n}{n^2+1}\right\}_{n=1}^{\infty}$

(*d*) $\left\{(1+2n)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ (*e*) $\left\{\dfrac{\ln\sqrt{n}}{n}\right\}_{n=1}^{\infty}$ (*f*) $\left\{(\sqrt{n})^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ (*g*) $\left\{\left(1-\dfrac{1}{n}\right)^{\frac{n}{2}}\right\}_{n=1}^{\infty}$

12.2 *Terminology for Sequences*

There are a few more basic terms that will be used to describe sequences, terms which are similar to those used for more general functions.

DEFINITION 12.2.1. A sequence $\{a_n\}_{n=1}^{\infty}$ is **non-decreasing** if each term is at least a big as its predecessor: $a_{n+1} \geq a_n$ for all n. Similarly, it is **non-increasing** if $a_{n+1} \leq a_n$ for all n. A sequence that is either non-decreasing or non-increasing is said to be **monotonic**.

Three simple examples illustrate the idea:

$$\{a_n\}_{n=1}^{\infty} = \left\{1+\frac{1}{n}\right\}_{n=1}^{\infty} = \{2, 3/2, 4/3, 5/4, \dots\}$$

is non-increasing while

$$\{b_n\}_{n=1}^{\infty} = \left\{1-\frac{1}{n}\right\}_{n=1}^{\infty} = \{0, 1/2, 2/3, 3/4, 4/5, \dots\}$$

is non-decreasing. The sequence

$$\{c_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, \dots\}$$

is neither non-increasing nor non-decreasing, so it is not monotonic.

YOU TRY IT 12.5. Look back to the sequences that were plotted beginning of this section and pick out the the non-increasing and the non-decreasing ones and those that were not monotonic.

One method to verify that a sequence $\{a_n\}_{n=1}^{\infty}$ is monotonic is to make use of the derivative function. Suppose that $a_n = f(n)$ for all n and the corresponding function $f(x)$ is differentiable on $[1,\infty)$. If the derivative $f'(x) \geq 0$ for all x in $[1,\infty)$, then both the function $f(x)$ and the corresponding sequence $\{a\}_{n=1}^{\infty}$ are non-decreasing. Likewise, if the derivative $f'(x) \leq 0$ for all x in $[1,\infty)$, then both the function $f(x)$ and the corresponding sequence $\{a\}_{n=1}^{\infty}$ are non-increasing. In either case the sequence is monotonic.

EXAMPLE 12.2.2. Consider the following sequences.

(*a*) $\{a_n\}_{n=1}^{\infty} = \left\{1+\dfrac{1}{n}\right\}_{n=1}^{\infty}$. Then the corresponding function is $f(x) = 1+\frac{1}{x}$. Notice that $f'(x) = -\frac{1}{x^2} < 0$ for all x in $[1,\infty)$. So the sequence is monotonic (decreasing).

(*b*) $\{a_n\}_{n=1}^{\infty} = \left\{\left(\dfrac{1}{2}\right)^n\right\}_{n=1}^{\infty}$. Then the corresponding function is $f(x) = \left(\frac{1}{2}\right)^x$. The derivative is $f'(x) = \ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x$. The natural log of any positive number smaller than 1 is negative. So $f'(x) < 0$ for all x in $[1,\infty)$. So the sequence is monotonic (decreasing).

(*c*) $\{a_n\}_{n=1}^{\infty} = \left\{\dfrac{\ln n}{n}\right\}_{n=1}^{\infty}$. Then the corresponding function is $f(x) = \frac{\ln x}{x}$. The derivative using the quotient rule is

$$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2}.$$

Since $\ln x > 1$ for $x > e$, we see that $f(x)$ is decreasing for $x > e$ and increasing for $x < e$. This translates to the sequence being decreasing for $n \geq 3$ and increasing for $n = 1,2$. So it is not monotonic. However, it is *eventually monotonic* since it is increasing for all $n \geq 3$.

DEFINITION 12.2.3. A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded** if there is some number M so that $|a_n| \leq M$ for all n.

The three simple sequences mentioned earlier are bounded. For

$$\{a_n\}_{n=1}^{\infty} = \left\{1+\frac{1}{n}\right\}_{n=1}^{\infty} = \{2, 3/2, 4/3, 5/4, \dots\}$$

notice $|a_n| \leq 2$. For

$$\{b_n\}_{n=1}^{\infty} = \left\{1-\frac{1}{n}\right\}_{n=1}^{\infty} = \{0, 1/2, 2/3, 3/4, 4/5, \dots\}$$

notice $|b_n| \leq 1$ and for

$$\{c_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, \dots\}$$

we have $|c_n| \leq 1$. Observe that we could have used larger bounds for each, e.g., $|a_n| \leq 12$.

From our perspective, the most important fact is that

THEOREM 12.2.4. If a sequence $\{a_n\}_{n=1}^{\infty}$ is both monotone and bounded, then it converges.

This is a hard theorem to prove and requires a more advanced understanding of the real numbers. Take Math 331.

As an example, the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ above were both monotone and bounded and both converge:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

while

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 - \frac{1}{n} = 1.$$