14.1 The Algebra of Series

There are several simple properties that apply to series that converge. We have already been using the first of these results.

THEOREM 14.1.1. Suppose that we have two convergent series: $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$. Then

(*a*) If c is a constant, then $\sum_{n=0}^{\infty} ca_n = cA$.

(*b*) $\sum_{n=0}^{\infty} a_n \pm b_n = A \pm B$.

EXAMPLE 14.1.2. Determine the sum of the series $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n + 3\left(-\frac{1}{2}\right)^n$ if it exists.

Solution. Using the Geometric Series Test (Theorem 13.2.2) and Theorem 14.1.1 we have

$$\begin{aligned}\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n + 3\left(-\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} 3\left(-\frac{1}{2}\right)^n \\ &= \frac{3}{1-\frac{1}{2}} + \frac{3}{1-(-\frac{1}{2})} \\ &= 6 + 2 = 8.\end{aligned}$$

14.2 The Divergence Test

The next theorem shows that for a series to converge, the terms of the series must get small as n gets large.

THEOREM 14.2.1 (The nth term Test). If $\sum_{n=0}^{\infty} a_n$ converges to A, then $\lim_{n\to\infty} a_n = 0$.

Proof. We make use of a comment we made earlier. Let S_n be the nth partial sum of the series. We know that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = A$ because both sequences are really just the same set of numbers. But

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

and

$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}.$$

So

$$S_n - S_{n-1} = a_n.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = A - A = 0,$$

which is what we wanted to prove. □

The theorem is seldom used in the form as stated above. Rather, if the nth term of a series does *not* converge to 0, then series cannot convergence. This is a more useful way of stating of Theorem 14.2.1 and in this form it is a test for *divergence*.

THEOREM 14.2.2 (The nth term test for divergence). If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Warning! The nth term test for divergence never allows us to conclude that a series converges, only that it does not converge. If $\lim_{n\to\infty} a_n = 0$, we can't conclude anything. The series may or may not converge. The test simply fails to provide us with any useful information in such a case.

EXAMPLE 14.2.3. Determine whether the series $\sum_{n=0}^{\infty} \frac{n^2+1}{3n^2+n+1}$ converges.

Solution. Notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{3n^2+n+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{3+\frac{1}{n}+\frac{1}{n}} = \frac{1}{3} \neq 0.$$

By the nth term test for divergence (Theorem 14.2.2), the series $\sum_{n=0}^{\infty} \frac{n^2+1}{3n^2+n+1}$ diverges.

EXAMPLE 14.2.4. Determine whether the series $\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$ converges.

Solution. This time using using one of our key limits (see Theorem 13.2)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{k}{n}\right)^n = e^k \neq 0.$$

By the nth term test for divergence (Theorem 14.2.2), the series $\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$ diverges.

EXAMPLE 14.2.5. Determine whether the series $\sum_{n=1}^{\infty} \sqrt[n]{n}$ converges.

Solution. Using another of our key limits (see Theorem 13.2)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = 1 \neq 0.$$

By the nth term test for divergence (Theorem 14.2.2), the series $\sum_{n=1}^{\infty} \sqrt[n]{n}$ diverges.

EXAMPLE 14.2.6. Determine whether the series $\sum_{k=1}^{\infty} \frac{2^n}{n^2} = \frac{2}{1} + \frac{4}{4} + \frac{8}{9} + \cdots$ converges.

Solution. Notice that both numerator and denominator both tend to infinity. So converting to x and using l'Hôpital's rule (twice!)

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)(\ln 2)2^x}{2} = \infty.$$

Use the derivative formula $\frac{d}{dx}(a^x) = a^x \ln a$, which is valid when $a > 0$.

By the nth term test for divergence (Theorem 14.2.2), the series diverges.

EXAMPLE 14.2.7. Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{n}$ and $\sum_{k=1}^{\infty} \frac{1}{n^2}$ converge.

Solution. Notice that both

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

and

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

In this situation the nth term test for divergence (Theorem 14.2.2) **fails** to provide us with any information. The series may or may not converge. In fact, we will soon see that one of these series converges while the other diverges.

EXAMPLE 14.2.8. Determine whether the series $\sum_{k=1}^{\infty} \cos \frac{1}{n}$ converges.

Solution. Notice that

$$\lim_{n\to\infty} \cos \frac{1}{n} = \cos 0 = 1 \neq 0.$$

By the nth term test for divergence (Theorem 14.2.2) the series diverges.

YOU TRY IT 14.5. Here are six series. Which of them can you say diverge by the nth term test for **divergence**? For which series is this test not helpful. Explain.

(a) $\sum_{n=1}^{\infty} \frac{3n+1}{2n+5}$ (b) $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ (c) $\sum_{n=1}^{\infty} (1.1)^n$

(d) $\sum_{n=1}^{\infty} \left(1+\frac{4}{n}\right)^n$ (e) $\sum_{n=1}^{\infty} \sqrt[n]{n}$ (f) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

WEBWORK: Click to try Problems 147 through 148. Use GUEST login, if not in my course.

14.3 *Series and Integrals: The Integral Test*

The integral test is great! It combines a number of key concepts in the course: Riemann sums, improper integrals, sequences, and series. Yet it is a very intuitive result.

Here's the idea: Say we have a series $\sum_{n=1}^{\infty} a_n$. Now we know that a_n is really a function $f(n)$ defined on the positive integers. Assume that the corresponding function $f(x)$ of the continuous variable x on the interval $[1, \infty)$ is **positive**, **continuous**, and **decreasing** and that the improper integral $\int_1^{\infty} f(x)\,dx$ **diverges**. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ has the corresponding function $f(x) = \frac{1}{x}$ which is positive, continuous, and decreasing for $x \geq 1$ and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges by the p-Power theorem.

In Figure 14.1 below, each • indicates a point of the sequence $\{a_n\}$ with the graph of the corresponding function $f(x)$ for $x \geq 1$. Notice that $f(x)$ is positive, decreasing, and continuous.

Now suppose that we approximate the area under $f(x)$ on the interval $[1, n+1]$ by using a **left**-hand Riemann sum, Left(n) with $\Delta x = \frac{(n+1)-1}{n} = 1$. The resulting rectangles, which are drawn in the picture, will extend indefinitely out to the right as n gets large. Each rectangle continues to have width 1. Notice that the height of the first rectangle is $f(1) = a_1$. Similarly, the second rectangle has height $f(2) = a_2$. More generally, the kth rectangle which lies above the interval $[k, k+1]$ has height $f(k) = a_k$. So the Riemann sum is

$$\text{Left}(n) = \sum_{k=1}^{n} f(k)\Delta x = \sum_{k=1}^{n} a_k \cdot 1 = \sum_{k=1}^{n} a_k = S_n.$$

That is, the left-hand Riemann sum is just the nth partial sum of the series. **Wow!**

$$S_n = \sum_{k=1}^{n} a_k = \text{Left}(n).$$

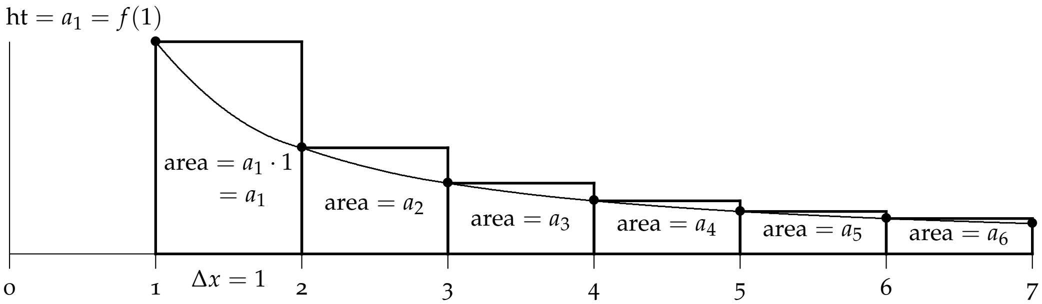


Figure 14.1: Each • indicates a point of the sequence $\{a_n\}$ with the graph of the corresponding function $f(x)$ for $x \geq 1$. Notice that $f(x)$ is positive, decreasing, and continuous. Because $f(x)$ is **decreasing**, the left-hand sum is an overestimate of the integral. And the function is integrable because it is **continuous**.

Notice that because $f(x)$ is **decreasing**, the left-hand sum is an overestimate of the integral. And the function is integrable because it is **continuous**. So

$$S_n = \sum_{k=1}^{n} a_k = \text{Left}(n) \geq \int_1^{n+1} f(x)\,dx.$$

Taking the limit as $n \to \infty$, we get the improper integral which we assumed diverged:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n \geq \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = \int_1^{\infty} f(x)\,dx \to \infty.$$

Since the series is at least as big as the corresponding improper integral:

Take-home message: If the improper integral **diverges** to infinity, so does the series!!

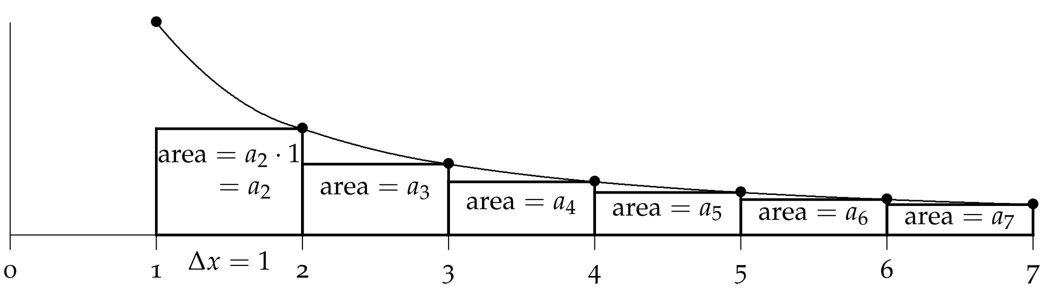
The Same Idea with a Twist

Assume, as before, that we have $\sum_{n=1}^{\infty} a_n$ and a function $f(x)$ that corresponds to the terms a_n and that $f(x)$ is **positive**, **continuous**, and **decreasing** for $x \geq 1$. This time assume that $\sum_{n=1}^{\infty} a_n$ diverges. If we toss out the first term, $\sum_{n=2}^{\infty} a_n$ still diverges.

This time approximate the area under $f(x)$ on the interval $[1, n+1]$ by using a **right**-hand Riemann sum, Right(n) sum. As before $\Delta x = 1$. The height of the first rectangle is $f(2) = a_2$. The second rectangle has height $f(3) = a_3$. In general, the kth rectangle which lies above the interval $[k, k+1]$ has height $f(k) = a_{k+1}$. So the Riemann sum is

$$\text{Right}(n) = a_2 \cdot \Delta x + a_3 \cdot \Delta x + \cdots + a_{n+1} \cdot \Delta x = a_2 \cdot 1 + a_3 \cdot 1 + \cdots + a_n \cdot 1 + a_{n+1} \cdot 1 = \sum_{k=2}^{n+1} a_k = T_n,$$

which is the $n+1$st partial sum of the series $\sum_{n=2}^{\infty} a_n$.



Because $f(x)$ is **decreasing**, the right-hand sum is an underestimate of the integral. So

$$T_n = \sum_{k=2}^{n+1} a_k = \text{Right}(n) \leq \int_1^{n+1} f(x)\,dx.$$

Taking the limit as $n \to \infty$, we get the entire series, which we assumed diverged, and the improper integral:

$$\infty = \lim_{n\to\infty} \sum_{k=2}^{n+1} a_k \leq \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = \int_1^{\infty} f(x)\,dx.$$

Since the improper integral is at least as big as the diverging series, the integral must diverge. Consequently, if the integral **con**verges, the series cannot diverge. In other words,

Take-home message 2: Therefore, if the improper integral **converges**, so does the series!! We can combine the two take-home messages into the following neat theorem.

THEOREM 14.3.1 (The Integral Test)**.** Given $\sum_{n=1}^{\infty} a_n$ and a **positive**, **continuous**, and **decreasing** function $f(x)$ such that $f(n) = a_n$.

1. $\sum_{n=1}^{\infty} a_n$ diverges if and only if diverges $\int_1^{\infty} f(x)\,dx$ diverges. This is the same as saying
2. $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Note: It is sufficient if $f(x)$ is positive and decreasing on some interval of the form $[a, \infty)$ where $a > 1$. It is the infinite tail of the the improper integral or the series that determines convergence or divergence, not the first few terms.

EXAMPLE 14.3.2. Does $\sum_{n=1}^{\infty} \frac{1}{n^2+4} = \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \cdots$ converge?

Solution. The series is not geometric and the Divergence Test is inconclusive. We use the integral test. The corresponding function is $f(x) = \frac{1}{x^2+4}$ and is clearly positive and continuous on $[1, \infty)$. Notice that $f'(x) = \frac{-2x}{(4+x^2)^2} < 0$ for all x in $[1, \infty)$. So f is decreasing. The improper integral that we must evaluate is $\int_1^{\infty} \frac{1}{x^2+4}\,dx$ which you should recognize as an inverse tangent integral.

We could also say that f is decreasing because as x increases, the denominator increases, but the numerator is constant—which makes the function values smaller.

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+4}\,dx$$
$$= \lim_{b\to\infty} \tfrac{1}{2}\arctan(\tfrac{x}{2})\Big|_1^b = \lim_{b\to\infty} \tfrac{1}{2}\arctan(\tfrac{b}{2}) - \tfrac{1}{2}\arctan(\tfrac{1}{2}) = \tfrac{1}{2}[\tfrac{\pi}{2} - \arctan(\tfrac{1}{2})].$$

Since the integral converges, by Theorem 14.3.1 the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ also converges.

14.4 *Applying the Integral Test: p-Series*

The integral test shows why improper integrals are important.

EXAMPLE 14.4.1. Let's look back at the two series in Example 14.2.7. The first was

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is called the **harmonic series**. (You should know it by name!) Does the harmonic series converge?

The second series was $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Does it converge? In Example 14.2.7 we saw that the nth term test provided no help since the nth terms go to 0 in both cases.

Solution. We can use the integral test to decide about convergence. For the harmonic series the corresponding improper integral that we must evaluate is $\int_1^\infty \frac{1}{x}\,dx$. We know that the integrand $f(x) = \frac{1}{x}$ is continuous and positive—and it is decreasing on $[1, \infty)$. (Think about the graph or notice that the derivative is $f'(x) = -\frac{1}{x^2}$ is negative.) From the p-Power test that we proved in the section on improper integrals, $\int_1^\infty \frac{1}{x}\,dx$ diverges since $p = 1$. By Theorem 14.3.1 the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

In a similar fashion for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the corresponding improper integral is $\int_1^\infty \frac{1}{x^2}\,dx$. Check that the integrand $\frac{1}{x^2}$ i s continuous, positive, and decreasing—$f'(x) = -\frac{2}{x^3} < 0$ on $[1, \infty)$. This time the improper integral converges by the p-Power test since $p > 1$. By Theorem 14.3.1 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

That was easy!

The two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ in Example 14.2.7 are examples of a particular type of series which we now single out for special attention. Using the p-Power test, we can generalize this result in the following way.

DEFINITION 14.4.2. If p is a real number, we say that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a ***p*-series**.

The main question is for which values of p does a p-series converge. Stop! You should be able to give the answer to this question! What is it?

THEOREM 14.4.3 (p-series Test). The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1. converges if $p > 1$
2. diverges if $p \leq 1$.

Proof. By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\int_1^\infty \frac{1}{x^p}\,dx$ converges. By the p-Power test, $\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$. Easy! □

EXAMPLE 14.4.4. The p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5}}$ converges because $p = \frac{5}{3} > 1$ while the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3}}$ diverges because $p = \frac{3}{5} \leq 1$.

Be on the lookout: Comparison of a series to something that is known to converge or diverge will be important in another day or two as we look at more complicated series.

More Examples of the Integral Test

The integral test applies to lots more series than p-series.

EXAMPLE 14.4.5. Does $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ converge?

Solution. We use the integral test. The corresponding function is $f(x) = \frac{1}{(x+1)\ln(x+1)}$ and is positive, decreasing (as x gets bigger, so does the denominator), and continuous. The improper integral that we must evaluate is $\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx$. Using a u-substitution with $u = \ln(x+1)$ and $du = \frac{1}{x+1}\,dx$ we find

$$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{(x+1)\ln(x+1)}\,dx$$
$$= \lim_{b\to\infty} \ln|\ln(x+1)|\Big|_1^b = \lim_{b\to\infty} \ln|\ln(b+1)| - \ln(\ln(2)) = +\infty.$$

By Theorem 14.3.1 the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ also diverges.

EXAMPLE 14.4.6. Does $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converge?

Solution. This is one that we could do as a telescoping series as in Example 13.1.3. Instead we use the integral test. The corresponding function is $f(x) = \frac{1}{x^2+x}$ and is clearly positive, decreasing, and continuous. The improper integral that we must evaluate is $\int_1^{\infty} \frac{1}{x^2+x}\,dx = \int_1^{\infty} \frac{1}{x} - \frac{1}{x+1}\,dx$ (check that the partial fractions are correct).

$$\int_1^{\infty} \frac{1}{x} - \frac{1}{x+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x} - \frac{1}{x+1}\,dx$$
$$= \lim_{b\to\infty} (\ln|x| - \ln|x+1|)\Big|_1^b$$
$$= \lim_{b\to\infty} \ln\left|\frac{x}{x+1}\right|\Big|_1^b$$
$$= \lim_{b\to\infty} \ln\frac{b}{b+1} - \ln\tfrac{1}{2} = \ln 1 - \ln\tfrac{1}{2} = \ln 2.$$

Since the integral converges, by Theorem 14.3.1 the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ also converges. Notice that unlike in Example 13.1.3, we do not know what $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges to, only that it does converge.

EXAMPLE 14.4.7. Does $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

Solution. The corresponding function is $f(x) = \frac{\ln x}{x}$. Notice that $f(1) = 0$ so it is not always positive on $[1, \infty)$. Further, we know that a function is decreasing when its derivative is negative. But using the quotient rule,

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \iff 1 < \ln x \iff x > e.$$

So f is positive, decreasing, and continuous only if $x > e$. That's good enough! It is the infinite tail of the series that matters, not the first few terms. Applying the integral test

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \frac{(\ln x)^2}{2}\Big|_1^b = \lim_{b\to\infty} \frac{(\ln b)^2}{2} - 0 = \infty$$

Since the integral diverges, by Theorem 14.3.1 the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges.

EXAMPLE 14.4.8. Does $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converge?

Solution. The nth term test fails and it is not a geometric or p-series. We use the integral test. The corresponding function is $f(x) = \frac{x}{e^x}$ and is clearly positive and continuous. It is decreasing since $f'(x) = \frac{e^x - xe^x}{(e^x)^2} = \frac{1-x}{e^x} \leq 0$ when $x \geq 1$. We evaluate (using integration by parts with $u = x$ and $dv = e^{-x}$)

$$\begin{aligned}
\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx &= \lim_{b\to\infty} -xe^{-x} + \int_1^b e^{-x}\,dx \\
&= \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_1^b \\
&= \lim_{b\to\infty} -\frac{b}{e^b} - \frac{1}{e^b} + \frac{1}{e} + \frac{1}{e} \\
&\overset{\text{l'Ho}}{=} \lim_{b\to\infty} -\frac{1}{e^b} + \left(0 + \frac{1}{e} + \frac{1}{e}\right) = \frac{2}{e}.
\end{aligned}$$

Since the integral converges, by Theorem 14.3.1 the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ also converges.

WEBWORK: Click to try Problems 149 through 152. Use GUEST login, if not in my course.

YOU TRY IT 14.6. Use the integral test, geometric series test, or the p-series test to determine whether the following series converge or diverge. While you could use the integral test on all of these, it is possible to use one of the other tests for four of the seven.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (d) $\sum_{n=1}^{\infty} \frac{1}{9+n^2}$

(e) $\sum_{n=1}^{\infty} \frac{1}{n^2+7n+10}$ (f) $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ (g) $\sum_{n=1}^{\infty} \frac{3n}{n^2+7}$ (h) $\sum_{n=1}^{\infty} \frac{2}{n^{1.00001}}$