14.5 *The Ratio Test*

The integral test is very powerful, but it is quite time consuming to do properly. One needs to very that the function corresponding to the series is positive, continuous, and decreasing (eventually). Then you need to do an improper integral! Moreover, there are lots of functions that are hard to integrate.

The ratio test does not depend on such knowledge—it is a self-contained or self-referential test and the results depend only on the series under consideration. One of its drawbacks, however, is that the test is someitmes inconclusive in terms of deciding whether or not there is convergence. However, we will see that this test is extremely useful in dealing with so-called power series, which is our final topic of the term. Ok, here's the test.

THEOREM 14.5.1 (The Ratio Test). Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

Proof. Here's the idea for the proof of part 1. Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. This means that when n is sufficiently large (say $n > k$), we have

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &\approx r \Rightarrow a_{n+1} \approx a_n r \\
\frac{a_{n+2}}{a_{n+1}} &\approx r \Rightarrow a_{n+2} \approx a_{n+1} r \approx a_n r^2 \\
\frac{a_{n+3}}{a_{n+2}} &\approx r \Rightarrow a_{n+3} \approx a_{n+2} r \approx a_n r^3 \\
\vdots &\approx \vdots \\
a_{n+k} &\approx a_n r^k
\end{aligned}$$

So the tail of the series is approximately geometric:

$$\sum_{k=0}^{\infty} a_{n+k} \approx \sum_{k=0}^{\infty} a_n r^k.$$

But the geometric series converges since $r < 1$, hence so should $\sum_{n=1}^{\infty} a_n$. □

Using the Ratio Test

The real utility of this test is that one need not know about another series to determine whether the series under consideration converges. This is very different from the comparison tests that we will discuss in later sections or the integral test which require knowledge beyond just the terms in the series.

EXAMPLE 14.5.2. Determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

Solution. This is not a geometric series, and it would be hard to integrate. But the terms are positive so let's try the ratio test.

ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2} < 1.$$

By the ratio test the series converges. That was quick!

☞: **The ratio test is very easy to use with both factorials and exponentials (powers) because there will be a lot of cancelation.**

EXAMPLE 14.5.3. Determine whether $\sum_{n=1}^{\infty} \frac{3^{n+2}n!}{4^n}$ converges.

Solution. There are both exponentials and factorials and the terms are positive, so let's try the ratio test. To eliminate compound fractions we can simplify the limit expression by multiplying by the reciprocal of a_n instead of dividing by it.

ARGUMENT:

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+2}(n+1)!}{4^{n+1}} \cdot \frac{4^n}{3^{n+1}n!} = \lim_{n\to\infty} \frac{3(n+1)}{4} = \infty.$$

By the ratio test the series diverges. That was quick!

EXAMPLE 14.5.4. Here's a very similar one: Determine whether $\sum_{n=1}^{\infty} \frac{8^{n+1}n^2}{2^{2n}}$ converges.

Solution. Because of the exponentials let's try the ratio test.

ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{8^{n+2}(n+1)^2}{2^{2n+2}} \cdot \frac{2^n}{8^{n+1}n^2} = \lim_{n\to\infty} \frac{8(n+1)^2}{2^2n^2} = \lim_{n\to\infty} \frac{2(n^2+2n+1)}{n^2}$$
$$\overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^2}{n^2} = 2 > 1.$$

By the ratio test the series diverges.

EXAMPLE 14.5.5. Determine whether $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges.

Solution. Because of the exponentials let's try the ratio test.

ARGUMENT: The terms are positive and

$$\begin{aligned} r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n} = \lim_{n\to\infty} \frac{3n^n}{(n+1)^{n+1}} &= \lim_{n\to\infty} \frac{3}{n+1} \cdot \frac{n^n}{(n+1)^n} \\ &= \lim_{n\to\infty} \frac{3}{n+1}\left(\frac{n}{n+1}\right)^n \\ &= \lim_{n\to\infty} \frac{3}{n+1}\left(\frac{1}{1+\frac{1}{n}}\right)^n \\ &= \lim_{n\to\infty} 0 \cdot \frac{1}{e} = 0 \end{aligned}$$

By the ratio test the series converges.

Why is the test inconclusive when the ratio is 1? The next example shows why.

EXAMPLE 14.5.6. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the p-series test we know that the former diverges while the later converges. But notice what happens when we try to apply the ratio test to each.

With the harmonic series we find

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{n} = 1.$$

With the second series we also get

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^2}{n^2} = 1.$$

Both ratios are 1, yet the first series diverges and the other converges (p-series test). For this reason the ratio test is inconclusive when the limit is 1.

Let's look at a few more examples.

EXAMPLE 14.5.7. Determine whether $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$ converges.

Solution. Because of the exponential and factorial let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n} = \lim_{n\to\infty} \frac{2}{(2n+1)(2n+2)} = 0 < 1.$$

By the ratio test the series converges.

EXAMPLE 14.5.8. Here's a slightly more complicated one: Determine whether $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converges.

Solution. Because of the factorials let's try the ratio test.
ARGUMENT: The terms are positive and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{(2n+1)2}{(n+1)}$$
$$\overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{4n}{n} = 4 > 1.$$

By the ratio test the series diverges.
WEBWORK: Click to try Problems 159 through 166. Use GUEST login, if not in my course.

14.6 *The Root Test*

The root is similar to the ratio test, though a bit less useful. It is a good test to use with series that contain powers but not so useful for series with factorials. The set up is essentially the same.

THEOREM 14.6.1 (The Root Test). Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \sqrt[n]{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

Proof. Here's the rough idea. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = r$. This means that when n is sufficiently large (say $n > k$), we have $a^n \approx r^n$ So the tail of the series is approximately geometric:

$$\sum_{k=n}^{\infty} a_k \approx \sum_{k=n}^{\infty} r^k.$$

But a geometric series converges if and only if $r < 1$. If the tail of the series converges, so does the entire series. So the entire series $\sum_{n=1}^{\infty} a_n$ will converge if $r < 1$ and certainly diverge if $r > 1$. When $r = 1$, the test turns out to be inconclusive, as we will see. □

Using the Root Test

The real utility of this test is that one need not know about another series to determine whether the series under consideration converges. Again, this is very different than with the comparison tests where some knowledge of another series is required.

EXAMPLE 14.6.2. Determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

Solution. This is not a geometric series, but the terms are positive and involve powers, so let's try the root test. (We just did this by the ratio test. Compare the arguments.)
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{1}{2},$$

using a key limit. By the root test the series converges. That was still pretty easy.

That was quick! The root test is very easy to use with exponentials (powers) because there will be a lot of cancelation.

EXAMPLE 14.6.3. Here's a another we did with the ratio test: Determine whether $\sum_{n=1}^{\infty} \frac{8^{n+1}n^2}{2^{2n}}$ converges.

Solution. Because of the powers let's try the root test.
ARGUMENT: The terms are positive and

$$\begin{aligned} r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{8^{n+1}n^2}{2^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{8^{n+1}n^2}}{\sqrt[n]{2^{2n}}} &= \lim_{n\to\infty} \frac{8^{(n+1)/n}n^{2/n}}{2^{2n/n}} \\ &= \lim_{n\to\infty} \frac{8^{1+1/n}(n^{1/n})^2}{2^2} \\ &= \lim_{n\to\infty} \frac{8\cdot 1}{4} \\ &= 2 > 1. \end{aligned}$$

By the root test the series diverges. Was this a bit more of a pain than doing it by the ratio test?

Why is the test inconclusive when the root is 1? The next example shows why.

EXAMPLE 14.6.4. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the p-series test we know that the former diverges while the later converges. But notice what happens when we try to apply the root test to each.

With the harmonic series we find

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{\sqrt[n]{1}}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

With the second series we also get

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \frac{\sqrt[n]{1}}{[\sqrt[n]{n}]^2} = \frac{1}{1^2} = 1.$$

Both roots are 1, yet the first series diverges and the other converges (p-series test). For this reason the root test is inconclusive when the limit is 1.

EXAMPLE 14.6.5. Determine whether $\sum_{n=1}^{\infty} \left(\frac{2n^3+1}{6n^3+n+2}\right)^{3n}$ converges.

Solution. Because of the power let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+1}{6n^3+n+2}\right)^{3n}} = \lim_{n\to\infty} \left(\frac{2n^3+1}{6n^3+n+2}\right)^{3} \overset{\text{HPwrs}}{=} \left(\frac{1}{3}\right)^3 < 1.$$

By the root test the series converges.

EXAMPLE 14.6.6. Determine whether $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^{n^2}$ converges.

Solution. Because of the power let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{2}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n^2/n} = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n} = e^2 > 1.$$

By the root test the series diverges.
WEBWORK: Click to try Problems 167 through 170. Use GUEST login, if not in my course.