14.8 Alternating Series

So far we've dealt primarily with series $\sum_{n=1}^{\infty} a_n$ where a_n was positive or at least non-negative. In fact the hypotheses of the integral test and the two comparison tests require that the terms of the series be positive. But some series have both positive and negative terms the simplest are so-called alternating series where the positive and negative terms alternate in the series. Such series can be defined as follows:

DEFINITION 14.8.1. An **alternating series** can be written in the form

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots \qquad \text{or} \qquad \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

where we assume that all the terms a_n are *positive*.

For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is called the **alternating harmonic series**. You should remember this series. We know that the harmonic series diverges, but what about the alternating harmonic series? Do the positive and negative terms cancel each other out in a way that the sum now converges? The answer is not obvious.

We have already encountered a few alternating series in the form of geometric series. For example, using the geometric series test (Theorem 13.2.2)

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{1}{1 + \frac{2}{3}} = \frac{3}{5},$$

which converges, while

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \cdots$$

diverges.

There's one key result for alternating series which we state without proof.

THEOREM 14.8.2 (The Alternating Series Test). Assume that $a_n > 0$ for all n. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if they satisfy the following two conditions:

1. $\lim_{n \to \infty} a_n = 0$
2. $a_{n+1} \leq a_n$ for all n (or at least eventually for all $n \geq k$).

Note: If an alternating series does not pass the first condition of the Alternating Series Test, then you can use the nth term test for divergence to conclude that the series actually *diverges*.

In other words, to show that an alternating series converges, it suffices to show that it satisfies the nth term test—that is, the nth term goes to 0,—and that the series is decreasing (or at least non-increasing). Note that the second condition can also be written as $\frac{a_{n+1}}{a_n} \leq 1$. The non-increasing condition only needs to be satisfied for the infinite tail of the series. It need not be true for the first few terms as long as *it is eventually true.* Let's look at some examples.

EXAMPLE 14.8.3 (The Alternating Harmonic Series). Does the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge?

Solution. Use the alternating series test. Here $a_n = \frac{1}{n}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ checks.
2. $\frac{1}{n+1} \leq \frac{1}{n}$ so $a_{n+1} \leq a_n$ for all n; the sequence is decreasing. Note: Another way to check that the series is decreasing is to examine the derivative of the associated function $f(x)$. Here $f(x) = \frac{1}{x}$, so $f'(x) = -\frac{1}{x^2} < 0$ for $x \geq 1$ and so is decreasing.

Since the series satisfies the two hypotheses, by Theorem 14.8.2 the alternating harmonic series converges *despite the fact the harmonic series diverges*.

EXAMPLE 14.8.4. Does $\sum_{n=1}^{\infty} \frac{n+1}{(-3)^n} = -\frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \cdots$ converge?

Solution. Use the alternating series test with $a_n = \frac{n+1}{3^n}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{3^n} = \lim_{x\to\infty} \frac{x+1}{3^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{1}{\ln(3)3^x} = 0$ checks.
2. $a_{n+1} \leq a_n \iff \frac{n+2}{3^{n+1}} \leq \frac{n+1}{3^n} \iff n+2 \leq \frac{3^{n+1}(n+1)}{3^n} \iff n+2 \leq 3n+3 \iff 0 \leq 2n+1$ which is always true for $n \geq 1$. The sequence is decreasing. Or check by taking the derivative of $f(x) = \frac{x+1}{3^x}$.

$$f'(x) = \frac{1 \cdot 3^x - (x+1)3^x \ln 3}{(3^x)^2} = \frac{1-(x+1)\ln 3}{3^x} < 0$$

for $x > 1$ and so is decreasing.

Here we have used the derivative rule $\frac{d}{dx}(a^x) = \ln(a)a^x$, where $a > 0$. Notice that this rule gives the correct answer for the derivative of the exponential function: $\frac{d}{dx}(e^x) = \ln(e)e^x = e^x$.

Since the series satisfies the two hypotheses, by Theorem 14.8.2 the alternating harmonic series converges.

EXAMPLE 14.8.5. Does $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1} = -\frac{1}{3} + \frac{4}{5} - \frac{9}{10} + \cdots$ converge?

Solution. Use the alternating series test with $a_n = \frac{n^2}{n^2+1}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \neq 0.$

Since the first hypothesis is not satisfied, the alternating series test does not apply. In this case the series diverges since the nth term does not go to 0.

Remember: If an alternating series does not pass the first condition of the Alternating Series Test, then you can use the nth term test for divergence to conclude that the series actually *diverges*.

EXAMPLE 14.8.6. Does $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$ converge?

Solution. Use the alternating series test with $a_n = \frac{n!}{(2n)!}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{(2n)!} = \lim_{n\to\infty} \frac{1 \cdot 2 \cdots n}{1 \cdot 2 \cdots n \cdot (n+1) \cdot (n+2) \cdots (2n)} = \lim_{n\to\infty} \frac{1}{(n+1)\cdots(2n)} = 0.$

2. $a_{n+1} \leq a_n \iff \frac{(n+1)!}{(2n+2)!} \leq \frac{n!}{(2n)!} \iff \frac{(n+1)!}{n!} \leq \frac{(2n+2)!}{(2n)!} \iff n+1 \leq (2n+1)(2n+2)$ which is always true for $n \geq 1$. Since the series satisfies the two hypotheses, by Theorem 14.8.2 the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{(2n)!}$ converges.

YOU TRY IT 14.8. Determine whether these series converge or diverge. Be sure to carefully **justify your answers with an argument**.

(*a*) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{n+2}$ (*b*) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n+1}$ (*c*) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ (*d*) $\sum_{n=0}^{\infty}(-1)^n \frac{2n}{n^2+1}$

WEBWORK: Click to try Problems 171 through 178. Use GUEST login, if not in my course.