14.9 Absolute and Conditional Convergence

Sometimes series have both positive and negative terms but they are not perfectly alternating like those in the previous section. For example

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \approx \frac{0.841}{1} + \frac{0.909}{4} + \frac{0.141}{9} - \frac{0.757}{16} - \frac{0.279}{25} + \frac{0.657}{36} + \cdots$$

is not alternating but does have both positive and negative terms.

So how do we deal with such series? The answer is to take the absolute value of the terms. This turns the sequence into a non-negative series and now we can apply many of our previous convergence tests. For example if we take the absolute value of the terms in the series above, we get

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|.$$

Since $|\sin n| \leq 1$, then

$$0 < \left| \frac{\sin n}{n^2} \right| < \frac{1}{n^2}.$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test ($p = 2 > 1$), so $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ converges by comparison. But what about the original series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$? The next theorem provides the answer: The series does converge.

THEOREM 14.9.1 (The Absolute Convergence Test). If $\sum_{n=1}^{\infty} |a_n|$ converges so does $\sum_{n=1}^{\infty} a_n$.

Proof. Given $\sum_{n=1}^{\infty} |a_n|$ converges. Define a new series $\sum_{n=1}^{\infty} b_n$, where

$$b_n = a_n + |a_n| = \begin{cases} a_n + a_n = 2a_n, & \text{if } a_n \geq 0 \\ a_n - a_n = 0, & \text{if } a_n < 0 \end{cases} \geq 0.$$

So $0 \leq b_n = a_n + |a_n| \leq |a_n| + |a_n| = 2|a_n|$. But $\sum_{n=1}^{\infty} 2|a_n|$ converges, hence by direct comparison $\sum_{n=1}^{\infty} b_n$ converges. Therefore

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} [(a_n + |a_n|) - |a_n|] = \sum_{n=1}^{\infty} b_n - |a_n| = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|$$

converges since it is the difference of two convergent series. □

Important Note. The converse is not true. If $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} |a_n|$ may or may not converge. For example, the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but if we take the absolute value of the terms, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This leads to the following definition.

DEFINITION 14.9.2. $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.
$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

EXAMPLE 14.9.3. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence. $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt[3]{n^2}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a p-series with $p = \frac{2}{3} \leq 1$. So the series of absolute values diverges. The original series is not absolutely convergent.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_n = \frac{1}{n^{2/3}}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{2/3}} = 0$.
2. Further $a_{n+1} \leq a_n$ because $\frac{1}{(n+1)^{2/3}} < \frac{1}{n^{2/3}}$.

Since the two conditions of the alternating series test are satisfied, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ is *conditionally convergent* by the alternating series test.

EXAMPLE 14.9.4. Determine whether $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely, conditionally, or not at all.

Solution. Notice that this series is not positive nor is it alternating since the first few terms are approximately

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \approx \frac{0.540}{1^2} - \frac{0.416}{2^2} - \frac{0.990}{3^2} + \frac{0.284}{4^2} + \cdots .$$

First we check absolute convergence. $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $p = 2 > 1$. We can use the direct comparison test. Since $0 \leq |\cos n| \leq 1$,

$$0 \leq \left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$$

for all n. Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ by the direct comparison test (Theorem 14.7.1). So the series of absolute values converges. The original series is absolutely convergent. We need not check further.

EXAMPLE 14.9.5. Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence. $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sqrt{n^2-1}}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$.
Notice that $\frac{1}{\sqrt{n^2-1}} \approx \frac{1}{n}$. So let's use the limit comparison test. The terms of the series are positive and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{n}{n} = 1 > 0.$$

Since the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series with $p = 1$), then $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2-1}} \right|$ diverges by the limit comparison test. So the series does not converge absolutely.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_n = \frac{1}{\sqrt{n^2-1}}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0.$

2. Further $a_{n+1} \leq a_n$ is decreasing because $\frac{1}{\sqrt{(n+1)^2-1}} < \frac{1}{\sqrt{n^2-1}}$. (You could also show the derivative is negative.) Since the two conditions of the alternating series test are satisfied, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$ is *conditionally convergent* by the alternating series test.

EXAMPLE 14.9.6. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n(2n^4+7)}{6n^9-2n}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence. $\sum_{n=1}^{\infty} \left| \frac{(-1)^n(2n^4+7)}{6n^9-2n} \right| = \sum_{n=1}^{\infty} \frac{2n^4+7}{6n^9-2n}$. Notice that $\frac{2n^4+7}{6n^9-2n} \approx \frac{1}{n^5}$. So let's use the limit comparison test. The terms of the series are positive and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^4+7}{6n^9-2n} \cdot \frac{n^5}{1} = \lim_{n\to\infty} \frac{2n^9+7n^5}{6n^9-2n} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^9}{6n^9} = \frac{1}{3} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges (p-series with $p = 5 > 1$), then $\sum_{n=1}^{\infty} \left| \frac{(-1)^n(2n^4+7)}{6n^9-2n} \right|$ converges by the limit comparison test. So the series converges absolutely.

EXAMPLE 14.9.7. Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence. $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$. We use the direct comparison test with $\frac{1}{n \ln n}$. Notice that $0 < \frac{1}{n \ln n} \leq \frac{1}{\ln n}$ because $n > 1$. Next $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (as we saw in earlier examples).[1] Consequently $\sum_{n=2}^{\infty} \left| \frac{1}{\ln n} \right|$ diverges by the direct comparison test. So the series does not converge absolutely.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_n = \frac{1}{\ln n}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$

2. Further a_n is decreasing since $f(x) = \frac{1}{\ln x} = (\ln x)^{-1}$ then $f'(x) = -\frac{(\ln x)^{-2}}{x} < 0$ for $x \geq 2$.

Since the two conditions of the alternating series test are satisfied, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is *conditionally convergent* by the alternating series test.

[1] To check that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, use the integral test and u-substitution with $u = \ln x$. $\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \ln|\ln x|\Big|_2^b = \lim_{b\to\infty} \ln|\ln b| - \ln(\ln 2) = +\infty$.

EXAMPLE 14.9.8. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2n}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence. $\displaystyle\sum_{n=1}^{\infty}\left|\frac{2^n}{2n}\right| = \sum_{n=2}^{\infty}\frac{2^n}{2n}$. Hey wait... does this series diverge? Use the nth term test:

$$\lim_{n\to\infty}\frac{2^n}{2n} = \lim_{x\to\infty}\frac{2^x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{2^x\ln 2}{2} = \infty.$$

Since $\lim_{n\to\infty} a_n \neq 0$ the series automatically diverges and cannot converge absolutely or conditionally.

Bonus Fact: The Ratio Test Extension

When we test for absolute convergence using the ratio test, we can say more. If the ratio r is actually greater than 1, the series will diverge. We don't even need to check conditional convergence.

THEOREM 14.9.9 (The Ratio Test Extension). Assume that $\displaystyle\sum_{n=1}^{\infty} a_n$ is a series with **non-zero** terms and let $r = \displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

1. If $r<1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges *absolutely*.
2. If $r>1$ (including ∞), then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r=1$, then the test is inconclusive. The series may converge or diverge.

This is most helpful when the series diverges. It says we can check for absolute convergence and if we find the absolute value series diverges, then the original series diverges. We don't have to check for conditional convergence. Huzzah!

EXAMPLE 14.9.10. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n!}{3^n}$ converges absolutely, conditionally, or not at all.

Solution. Here's a perfect place to use the ratio test extension because there is a factorial.

$$r = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!}{3^{n+1}}\cdot\frac{3^n}{(-1)^n n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \infty.$$

The (original) series diverges by the ratio test extension. That was easy!! The ratio test extension says we don't have to check for conditional convergence.

EXAMPLE 14.9.11. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)!}{2^n n}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence using the ratio test because of the factorial.

$$r = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)!}{2^{n+1}(n+1)}\cdot\frac{2^n n}{(-1)^n(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(n+2)n}{2(n+1)}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n}{2n+2}\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{n}{2} = \infty.$$

The (original) series diverges by the ratio test extension.

EXAMPLE 14.9.12. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{n(-2)^n}{3^{n+1}}$ converges absolutely, conditionally, or not at all.

Solution. Check absolute convergence using the ratio test extension.

$$r = \lim_{n\to\infty} \left| \frac{(n+1)(-2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(-2)^n} \right| = \lim_{n\to\infty} \left| \frac{2(n+1)}{3n} \right| \stackrel{\text{HPwrs}}{=} \frac{2}{3} < 1.$$

The (original) series converges absolutely by the ratio test extension.

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