Integration by Substitution

In this chapter we expand our methods of antidifferentiation. We have encountered integrals which we have been unable to determine because we did not know an antiderivative for the integrand. The technique we discuss below is simply reversing the chain rule for derivatives. Remember, every derivative rule can be reversed to create an antidifferentiation rule. Our objectives are to

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.

All are techniques for integrating composite functions. The discussion is split into two parts: pattern recognition and change of variables. Both techniques involve a so-called u-substitution. With pattern recognition we perform the substitution mentally, and with change of variables we write the substitution in detailed steps.

Pattern Recognition

Recall that the *chain rule* for derivatives of composite functions states:

THEOREM 3.1.1 (Chain Rule)**.** If $y = F(u)$ and $u = g(x)$ are differentiable then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x)$$

or, equivalently,

$$\frac{d}{dx}\left[F(u)\right] = F'(u)\frac{du}{dx}.$$

Notice in this set-up that

$$\frac{du}{dx} = g'(x) \tag{3.1}$$

the corresponding differentials are[1]

$$du = g'(x)dx. \tag{3.2}$$

Reversing the chain rule, from the definition of an antiderivative, we get:

THEOREM 3.1.2 (Substitution)**.** Assume F is an antiderivative of f (so $F' = f$) and that g is differentiable. Then letting $u = g(x)$ so $du = g'(x)dx$, we get

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + c = F(g(x)) + c. \tag{3.3}$$

[1] Though $\frac{du}{dx}$ is a derivative and a single entity, when working with differentials it is as if one can multiply each side of (3.1) by dx to obtain the equation in (3.2).

Once we know F is and antiderivative of f, then by definition of antiderivative we have $\int f(u)\,du = F(u) + c$ ... for example, $\int \cos u\,du = \sin u + c$ *no matter what the differentiable function u is*. This is the whole point of substitution whether it is done mentally or written out: Convert the problem into one that we can solve 'just by looking at it.'

To apply Theorem 3.1.2 directly, we must recognize the presence of $f(g(x))$ and $g'(x)$, i.e., $f(u)$ and $\frac{du}{dx}$. Note that the composite function in the integrand has an *outside function f* and an *inside function g*. Moreover, the derivative $g'(x)$ must also be present as a factor of the integrand so that (3.3) becomes:

$$\int \overbrace{f}^{\text{outter}} (\underbrace{g(x)}_{\text{inner}}) \quad \overbrace{g'(x)}^{\text{inner deriv}} \quad dx.$$

EXAMPLE 3.1.3 (Recognizing the pattern). Determine $\displaystyle\int \cos(x^3)3x^2\,dx$.

Solution. There is a function which is literally 'inside' the parentheses in this problem. So the inside function is $u = x^3$ and its derivative, $3x^2$ is a factor in the integrand.

$$\int \cos(\overbrace{x^3}^{u})\overbrace{3x^2\,dx}^{du}.$$

In other words,[2] the integral can be thought of as

[2] Let $u = x^3$ and $du = 3x^2\,dx$.

$$\int \cos(u)\,du = \sin u + c = \sin(x^3) + c,$$

where we have been careful to rewrite the answer in terms of the original variable x. Can you do this manipulation in your head? You should be able to. Check the answer by differentiating.

EXAMPLE 3.1.4 (Recognizing the pattern). Determine $\displaystyle\int e^{3t}\,dt$.

Solution. We have seen this type of problem before... it requires a 'mental adjustment.' In our current thinking the 'obvious' inside function is $u = 3t$. The corresponding differential is $du = 3\,dt$, so we think $\frac{1}{3}\,du = dt$. Mentally

$$\int e^{\overbrace{3t}^{u}} \overbrace{dt}^{\frac{1}{3}\,du} = \int \tfrac{1}{3}e^u\,du = \tfrac{1}{3}e^u + c = \tfrac{1}{3}e^{3t} + c.$$

Mental adjustment is just a simple case of u-substitution which you should be able to do in your head. Check the answer by differentiating.

EXAMPLE 3.1.5 (Recognizing the pattern). Determine $\displaystyle\int e^{3\tan x}\sec^2 x\,dx$.

Solution. Before reading further, try to do the manipulation required in your head or with minimal scratch work.

Ok, let's work it out—we will write down what you should be thinking. Again there is an 'obvious'[3] inside function: $3\tan x$ and its derivative is $3\sec^2 x$. Notice that we have $\sec^2 x$ as a factor in the integrand, but we are missing a factor of 3. When we are only off by a *constant factor* we can make a mental adjustment and still do the integral. Think of $3\tan x$ as u so $3\sec^2 x\,dx$ is the differential (derivative) du, so in our integrand we have $\sec^2 x\,dx = \frac{1}{3}du$.

[3] Why is it obvious?

$$\int e^{\overbrace{3\tan x}^{u}} \overbrace{\sec^2 x\,dx}^{\frac{1}{3}du},$$

which is exactly the same as in Example 3.1.4. So the integral can be thought of as

$$\int \tfrac{1}{3}e^u\,du = \tfrac{1}{3}e^u + c = \tfrac{1}{3}e^{3\tan x} + c,$$

where we have been careful to rewrite the answer in terms of the original variable x. For this one, you might have to write out a note or two on the substitution to complete the problem, but in the end the pattern is no different than the previous one in Example 3.1.4. That's what substitution is all about—and that's what mathematics is all about. Recognizing patterns and taking advantage of them. Check the answer by differentiating.

EXAMPLE 3.1.6 (Recognizing the pattern). Determine $\int 12x^3\sqrt{3x^4+1}\,dx$.

Solution. Start by trying to do it in your head. Find the inside function u and and its differential du.

This time the inside function is $u = 3x^4 + 1$ and its differential is $du = 12x^3\,dx$, which is a factor in the integrand. So (mentally) reordering the integral and writing the square root as a power, we think

$$\int (\underbrace{3x^4+1}_{u})^{1/2}\cdot\overbrace{12x^3\,dx}^{du} = \int u^{1/2}\,du = \tfrac{2}{3}u^{3/2} + c = \tfrac{2}{3}(3x^4+1)^{3/2} + c.$$

Check the answer by differentiating.

EXAMPLE 3.1.7 (Recognizing the pattern). Determine $\int (4x^3+3x^2)^4(6x^2+3x)\,dx$.

Solution. Ok, this is a bit harder. Try to find the inside function u and and its differential du. Do we have both?

The inside function is $u = 4x^3 + 3x^2$ and[4] its differential is $du = (12x^2+6x)\,dx$. Notice that we have $\frac{1}{2}du = (6x^2+3x)\,dx$ as a factor in the integrand; we are off by a constant factor. Rethinking the integral we get

[4] Could we have let $u = 12x^2 + 6x$ since it, too, is inside a set of parentheses? Why or why not?

$$\int (\overbrace{4x^3+3x^2}^{u})^4\,\overbrace{(6x^2+3x)\,dx}^{\frac{1}{2}du} = \int \tfrac{1}{2}u^4\,du = \tfrac{1}{10}u^5 + c = \tfrac{1}{10}(4x^3+3x^2)^5 + c.$$

Check the answer by differentiating.

EXAMPLE 3.1.8 (Recognizing the pattern). Determine $\int \frac{6x}{x^2+1}\,dx$.

Solution. What do you think this time?

The 'inside' function is less obvious. However, if we try to spot the pieces u and du, it is pretty clear that that we want u to be the higher-degree term. If we let u be the denominator, $u = x^2+1$, then the differential $du = 2x\,dx$. Notice that we have $3du = 6x\,dx$ as the numerator in the integrand; we are off by a constant factor. Rethinking the integral we get

$$\int \frac{\overbrace{6x\,dx}^{3du}}{\underbrace{x^2+1}_{u}} = 3\int \frac{1}{u}\,du = 3\ln|u| + c = 3\ln(x^2+1) + c.$$

Check the answer by differentiating.

YOU TRY IT 3.1. Determine each of the following:

(a) $\int \sec^2(\sin x)\cos x\,dx$ (b) $\int \frac{2\ln t}{t}\,dt$ (c) $\int (x+2)\tan(x^2+4x)\,dx$

ANSWER TO YOU TRY IT 3.1. (a) $\tan(\sin x) + c$; (b) $(\ln t)^2 + c$; (c) $\frac{\ln(\sec(x^2+4x))}{2} + c$.

WEBWORK: Click to try Problems 50 through 58. Use GUEST login, if not in my course.

Change of Variables

The process of changing variables formalizes what we were doing in the previous section. I am imagining that you were trying to do the previous problems in your head. They get harder because we have to keep track of more things. If u is complicated, so is du. Do we have du exactly or are we off by a constant? Then the integral itself, even after substitution, may not be entirely obvious.

With a formal change of variables, we completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in the previous examples, it is useful for complicated integrands.

Let's try some. The first few will be similar to the last few examples to illustrate the process.

EXAMPLE 3.1.9. Determine $\int (3-2\sin x)^6 \cos x\,dx$.

Solution. The 'inside' function is literally inside the parentheses:

$$u = 3 - 2\sin x, \text{ so } du = -2\cos x\,dx$$
$$\text{or } -\frac{1}{2}du = \cos x\,dx.$$

Now, using $u = 3-2\sin x$ and $-\frac{1}{2}du = \cos x\,dx$ quite literally substitute or replace the pieces in the original integral. *Rewriting* (not just rethinking) the integral we get

$$\int (3-2\sin x)^6 \cos x\,dx = \int -\frac{1}{2}u^6\,du = -\frac{1}{14}u^7 + c = -\frac{1}{14}(3-2\sin x)^7 + c.$$

To keep track of the negative sign and constant factor in the integration is easier when you write down more details rather than trying to keep it all in your head. Check the answer by differentiating.

EXAMPLE 3.1.10. Determine $\int \frac{\sin x}{\cos^4 x}\,dx$.

Solution. This time the denominator is the more complicated function which is often an indication that it is the 'u'. So let

$$u = \cos x, \text{ so } du = -\sin x\,dx$$
$$\text{or } -du = \sin x\,dx.$$

Now replace the pieces in the original integral.

$$\int \frac{\sin x}{\cos^4 x}\,dx = \int -u^{-4}\,du = \frac{1}{3}u^{-3} + c = \frac{1}{3}(\cos x)^{-3} + c = \frac{1}{3\cos^3 x} + c.$$

Notice that there are two negative signs that are introduced along the way. One or the other would be easy to lose if you try to keep all the details in your head. Check the answer by differentiating.

EXAMPLE 3.1.11. Determine $\int \frac{3e^x + 3e^{-x}}{(e^x - e^{-x})^3}\,dx$.

Solution. Again the denominator is the more complicated function and the 'inside' function is literally inside the parentheses. So let

$$u = e^x - e^{-x}, \text{ so } du = e^x + e^{-x}\,dx$$
$$\text{or } 3\,du = (3e^x + 3e^{-x})\,dx.$$

Replace the pieces in the original integral.

$$\int \frac{3e^x + 3e^{-x}}{(e^x - e^{-x})^3}\,dx = \int 3u^{-3}\,du = -\tfrac{3}{2}u^{-2} + c = -\tfrac{3}{2}(e^x - e^{-x})^{-2} + c.$$

The problem ends up being very similar to the previous example. Check the answer by differentiating.

EXAMPLE 3.1.12. Determine $\int \tan x\,dx$.

Solution. We don't know an antiderivative for this integrand, and it does not look like a substitution problem. However, we can rewrite the integrand.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

We could let $u = \sin x$ and then $du = \cos x\,dx$. Notice though that we do not have $\cos x\,dx$; in our integrand we have $\frac{1}{\cos x}\,dx$. We are *not* off by a constant factor so we have to try something else. Let

$$u = \cos x, \text{ so } du = -\sin x\,dx$$
$$\text{or } -du = \sin x\,dx.$$

We saw this same substitution in Example 3.1.10. Now replace the pieces in the original integral.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln|cosx| + c = \ln|\sec x| + c.$$

Check the answer by differentiating.

The solution to Example 3.1.12 is important enough so that you should memorize it so that you do not have to re-invent the solution each time you see it.

$$\boxed{\int \tan x\,dx = \ln|\sec x| + c.} \tag{3.4}$$

YOU TRY IT 3.2. Fill in the blank with a function that makes this an easy problem to do and then solve the problem:

$$\int \tan(e^{x^3+1})\underline{\hspace{2cm}}\,dx.$$

ANSWER TO YOU TRY IT 3.2. $u = 3x^2$

EXAMPLE 3.1.13. Here's the other trig integral I want you to know. Determine $\int \sec x\,dx$.

Solution. Again this does not look like a substitution problem, even if you rewrite $\sec x$ as $\frac{1}{\cos x}$. I still remember my Calc II teacher in college showing us how to solve this integral using substitution. I thought it was very neat!

Ok, the trick here (and it is a trick) is to multiply the integrand by 1 in the form

$$\frac{\sec x + \tan x}{\sec x + \tan x}.$$

Now

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx.$$

How does this help? Now we can use substitution. This is one of the few times where the less complicated expression—the denominator—is actually u. Let

$$u = \sec x + \tan x, \text{ so } du = (\sec x \tan x + \sec^2 x)\,dx.$$

The numerator is precisely du. Now replace the pieces in the original integral.

$$\int \sec x\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln|\sec x + \tan x| + c.$$

Neat! Check the answer by differentiating.

The solution to Example 3.1.13 is also important enough so that you should memorize it so that you do not have to re-invent the solution each time you see it.

$$\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + c.} \tag{3.5}$$

EXAMPLE 3.1.14. The two new integration rules in (3.4) and (3.5) can themselves be used in u-substitution problems. Determine $\int 2x\sec(4x^2+1)\,dx$.

Solution. Ok, this time the inside function is

$$u = 4x^2 + 1 \text{ so } du = 8x\,dx \quad \text{or } \tfrac{1}{4}du = 2x\,dx.$$

Ok, now we can make use of (3.5): Now

$$\begin{aligned}\int 2x\sec(4x^2+1)\,dx = \int \tfrac{1}{4}\sec u\,du &= \tfrac{1}{4}\ln|\sec u + \tan u| + c \\ &= \tfrac{1}{4}\ln|\sec(4x^2+1) + \tan(4x^2+1)| + c.\end{aligned}$$

(Check the answer by differentiating.)

EXAMPLE 3.1.15 (Complex Pattern Recognition). Determine $\int \frac{x}{1+x^4}\,dx$.

Solution. When we try the 'obvious' substitution here, namely $u = 1 + x^4$, we see that $du = 4x^3\,dx$. However, we don't have a factor of x^3 in the integrand. If we step back a second, the integrand looks a bit like the derivative of an arctangent function. The denominator consists of 1 plus something squared. The 'something squared' is x^4. So suppose we let

$$\begin{aligned} u^2 = x^4, \text{ then } u = x^2, &\text{ so } du = 2x\,dx \\ &\text{or } \tfrac{1}{2}\,du = x\,dx.\end{aligned}$$

Now replace the pieces in the original integral.

$$\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}}{1+u^2}\,du = \tfrac{1}{2}\arctan u + c = \tfrac{1}{2}\arctan(x^2) + c.$$

Check the answer by differentiating.

EXAMPLE 3.1.16 (Complex Pattern Recognition). Determine $\int \frac{x^2}{\sqrt{1-4x^6}}\,dx$.

Solution. Again the 'obvious' substitution $u = 1 - 4x^6$ fails because $du = -24x^5\,dx$ and we don't have a factor of x^5 in the integrand. But could this be an arcsine problem? If the denominator is to have the right form it must look like $\sqrt{1-u^2}$. That would mean that $\sqrt{1-u^2} = \sqrt{1-4x^6}$. This means that $u^2 = 4x^6$, so we must have $u = 2x^3$. But if

$$\begin{aligned} u = 2x^3, &\text{ then } du = 6x^2\,dx \\ &\text{or } \tfrac{1}{6}\,du = x^2\,dx.\end{aligned}$$

Things have worked out: we do have an x^2 in the integrand! Now replace the pieces in the original integral.

$$\int \frac{x^2}{\sqrt{1-4x^6}}\,dx = \int \frac{\frac{1}{6}}{\sqrt{1-u^2}}\,du = \tfrac{1}{6}\arcsin u + c = \tfrac{1}{6}\arcsin(2x^3) + c.$$

Check the answer by differentiating.

EXAMPLE 3.1.17 (Complex Pattern Recognition). Determine $\int xe^{x^2}\tan(e^{x^2})\,dx$.

Solution. Here the 'obvious' substitution seems to be $u = e^{x^2}$ because this term is inside the parentheses. If

$$u = e^{x^2}, \text{ then } du = 2xe^{x^2}\,dx$$
$$\text{or } \frac{1}{2}\,du = xe^{x^2}\,dx.$$

Now replace the pieces in the original integral.

$$\int xe^{x^2}\tan(e^{x^2})\,dx = \int \frac{1}{2}\tan u\,du = \frac{1}{2}\ln|\sec u| + c = \frac{1}{2}\ln|\sec(e^{x^2})| + c.$$

Notice that we have used our work in (3.4) as well as substitution to solve the problem. Check the answer by differentiating.

WEBWORK: Click to try Problems 59 through 63. Use GUEST login, if not in my course.

Take-home Message

Right now you know antiderivative formulæ for a half-dozen or so types of functions: logs, exponentials, a few trig and inverse trig functions, and powers. These all have their corresponding u-substitution forms. You simply need to be on the lookout for them. Here's a list of the most important forms. Remember that u will vary from problem to problem and you may need to adjust by a constant.

(a) $\int \sin u\,du$	(b) $\int \cos u\,du$	(c) $\int \tan u\,du$	(d) $\int e^u\,du$
(e) $\int \frac{1}{u}\,du$	(f) $\int \frac{1}{1+u^2}\,du$	(g) $\int \frac{1}{\sqrt{1-u^2}}\,du$	(h) $\int u^k\,du$

More Complex Substitutions

This section consists of a few more examples where substitution is used, but may be harder to see. When you think about it for a second, at this point if you are faced with an integral, you have only three choices: (1) you recognize the integrand immediately as the derivative of one of a couple of handfuls of basic functions; (2) you try substitution; (3) you are stuck. As the term progresses we will develop additional integration techniques and you will have to become adept at sorting out which technique applies to a given situation. But for now, if the answer is not obvious, you have only one choice: Try substitution by identifying u and du in the integrand.

EXAMPLE 3.1.18 (Complex Pattern Recognition: Logs). Here's a classic: Determine $\int \frac{2}{x\ln x}\,dx$.

Solution. From the comments above, since this is not an integrand that you are likely to recognize immediately, we need to try substitution. In several instances we have used the denominator of an expression as u. If we try that here, $u = x \ln x \; du = (1 + \ln x)\, dx$ which does not appear in the integrand. Looking at the expressions as two pieces $\frac{1}{x}$ and $\frac{1}{\ln x}$ you should notice that $\frac{1}{x}$ is the derivative of $\ln x$. Let's see where this leads. If

$$u = \ln x, \text{ then } du = \frac{1}{x}\, dx$$
$$\text{or } 2\, du = \frac{2}{x}\, dx.$$

Now replace the pieces in the original integral.

$$\int \frac{2}{x \ln x}\, dx = \int \frac{2}{u}\, du = 2 \ln |u| + c = 2 \ln |\ln x| + c.$$

You should check the answer by differentiating.

YOU TRY IT 3.3. These problems are similar to Example 3.1.18. Try them now.

$$(a) \int \frac{\ln t}{t}\, dt \qquad (b) \int \frac{(\ln x)^3}{x}\, dx \qquad (c) \int \frac{1}{x(\ln x)^3}\, dx$$

ANSWER TO YOU TRY IT 3.3. (a) $\frac{(\ln x)^2}{2} + c$; (b) $\frac{(\ln x)^4}{4} + c$; (c) $-\frac{(\ln x)^{-2}}{2} + c$.

Complex Pattern Recognition: Inverse Trig Functions

Ok, I waffle back and forth on the most efficient and effective way to present this material. Let's plunge ahead and see where this takes us. In Examples 3.1.15 and 3.1.16 we used substitution with inverse trig functions. In each case we had an integrand that had the form $\frac{1}{1+u^2}$ or $\frac{1}{\sqrt{1-u^2}}$, where u was a function of x. The key was the "1" plus or minus something squared which reminded us of the arctangent or arcsine functions.

The problem is a bit more complicated if we have an integrand with something other than a 1. Here's what I mean: Assume $a > 0$. Suppose our integrand involves $\frac{1}{\sqrt{a^2-x^2}}$, where u is some function of x. With a bit of algebra, we can rewrite this is more convenient form:

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - \frac{a^2x^2}{a^2}} = \sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)} = a\sqrt{1 - \frac{x^2}{a^2}} \tag{3.6}$$

So now

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}}\, dx.$$

Now the 'obvious' substitution is $u^2 = \frac{x^2}{a^2}$ so that

$$u = \frac{x}{a}, \text{ then } du = \frac{1}{a}\, dx$$

We have a factor of $\frac{1}{a}$ in the integrand. Substituting

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}}\, dx = \int \frac{1}{\sqrt{1 - u^2}}\, du = \arcsin u + c = \arcsin \frac{x}{a} + c. \tag{3.7}$$

Since the name of the variable is irrelevant, we can write (3.7) more generally as

$$\boxed{\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \arcsin \frac{u}{a} + c.} \tag{3.8}$$

Using exactly the same sort of process, we find

$$\boxed{\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + c.} \tag{3.9}$$

Caution: First, note that you must have u and du exactly in the integrand to get the answers in (3.8) and (3.9). Most of the time, one will need to adjust by some constant factor. Second, observe that there is factor of $\frac{1}{a}$ in this antiderivative in (3.9).

Should you memorize these two formulæ? It will make your life easier if you do. The problem, of course, is that one has to remember that the arctangent has an additional factor of $\frac{1}{a}$.

YOU TRY IT 3.4. Derive equation (3.9) using a process similar to what we did to prove (3.8). Why is there a factor of $\frac{1}{a}$ in this antiderivative but not for the arcsine function in (3.8)

The next two examples show how (3.8) and (3.9) are used.

EXAMPLE 3.1.19 (Complex Pattern Recognition: Inverse Trig). Determine $\displaystyle\int \frac{1}{\sqrt{36-25x^2}}\,dx.$

Solution. Use (3.8). Here

Ordinary u-substitution fails because $u = 36-25x^2$ so $du = -50x\,dx$ and there is no 'x' term in the numerator.

$$\begin{aligned} a^2 = 36 &\text{ and } u^2 = 25x^2 \\ \text{or } a = 6, &\text{ and } u = 5x \\ &\text{so } du = 5\,dx \\ &\text{so } \frac{1}{5}\,du = dx. \end{aligned}$$

Now replace the pieces in the original integral.

$$\int \frac{1}{\sqrt{36-25x^2}}\,dx = \int \frac{\frac{1}{5}}{\sqrt{a^2-u^2}}\,du = \frac{1}{5}\arcsin\frac{u}{a} + c = \frac{1}{5}\arcsin\frac{5x}{6} + c.$$

Notice that we have had to adjust (3.8) by a constant factor of $\frac{1}{5}$ because we did not have du exactly in the integrand.

EXAMPLE 3.1.20 (Complex Pattern Recognition: Inverse Trig). Determine $\displaystyle\int \frac{x}{36+25x^4}\,dx.$

Solution. Use (3.9). Here

Ordinary u-substitution fails because $u = 36+25x^4$ so $du = 100x^3\,dx$ and there is no 'x^3' term in the numerator, only an 'x'.

$$\begin{aligned} a^2 = 36 &\text{ and } u^2 = 25x^4 \\ \text{or } a = 6, &\text{ and } u = 5x^2 \\ &\text{so } du = 10x\,dx \ \text{ so } \tfrac{1}{10}\,du = x\,dx. \end{aligned}$$

Now carefully replace the pieces in the original integral.

$$\int \frac{x}{36+25x^4}\,dx = \int \frac{\frac{1}{10}}{a^2+u^2}\,du = \frac{1}{10}\cdot\frac{1}{a}\cdot\arctan\frac{u}{a} + c = \frac{1}{60}\arctan\frac{5x^2}{6} + c.$$

Be careful! Not only did we have to adjust (3.9) by a constant factor of $\frac{1}{10}$ but we have to remember the additional factor of $\frac{1}{a} = \frac{1}{6}$ here giving us $\frac{1}{60}$ int the final answer.

EXAMPLE 3.1.21 (Complex Pattern Recognition: Inverse Trig). Determine $\displaystyle\int \frac{1}{2+9x^2}\,dx.$

Solution. Use (3.9). Here

$$\begin{aligned} a^2 = 2 &\text{ and } u^2 = 9x^2 \\ \text{or } a = \sqrt{2}, &\text{ and } u = 3x \\ &\text{so } du = 3\,dx \\ &\text{so } \frac{1}{3}\,du = dx. \end{aligned}$$

Now replace the pieces in the original integral and use (3.9).

$$\int \frac{1}{2+9x^2}\,dx = \int \frac{\frac{1}{3}}{a^2+u^2}\,du = \frac{1}{3}\cdot\frac{1}{a}\arctan\frac{u}{a}+c = \frac{1}{3\sqrt{2}}\arctan\frac{3x}{\sqrt{2}}+c.$$

Check the answer by differentiating.

YOU TRY IT 3.5. These problems are similar to Examples 3.1.20 and 3.1.21. Try them now.

(*a*) $\displaystyle\int \frac{5}{3+25t^2}\,dt$ (*b*) $\displaystyle\int \frac{1}{\sqrt{9-16x^2}}\,dx$

ANSWER TO YOU TRY IT 3.5.

(*a*) $\frac{1}{\sqrt{3}}\arctan\frac{5t}{\sqrt{3}}+c$ (*b*) $\frac{1}{4}\arcsin\frac{4x}{3}+c$.

EXAMPLE 3.1.22 (Complex Pattern Recognition). Determine $\displaystyle\int \frac{x}{2+9x^2}\,dx$.

Solution. Do you see what's going on here?[5] Don't get fooled by all the inverse trig problems we've been doing. This is simpler substitution leading to a log integral. Here

$$u = 2+9x^2 \text{ so } du = 18x\,dx$$
$$\text{so } \frac{1}{18}\,du = x\,dx.$$

Now replace the pieces in the original integral.

$$\int \frac{x}{2+9x^2}\,dx = \int \frac{\frac{1}{18}}{u}\,du = \frac{1}{18}\ln|u|+c = \frac{1}{18}\ln(2+9x^2)+c.$$

Check the answer by differentiating.

EXAMPLE 3.1.23 (Complex Pattern Recognition). Determine $\displaystyle\int \frac{x}{2+9x^4}\,dx$.

Solution. Now we are back to an inverse trig problem again. Here

$$a^2 = 2 \text{ and } u^2 = 9x^4$$
$$\text{or } a = \sqrt{2}, \text{ and } u = 3x^2$$
$$\text{so } du = 6x\,dx$$
$$\text{so } \frac{1}{6}\,du = x\,dx.$$

Replace the pieces in the original integral and use (3.9).

$$\int \frac{x}{2+9x^4}\,dx = \int \frac{\frac{1}{6}}{a^2+u^2}\,du = \frac{1}{6}\cdot\frac{1}{a}\arctan\frac{u}{a}+c = \frac{1}{6\sqrt{2}}\arctan\frac{3x^2}{\sqrt{2}}+c.$$

Check the answer by differentiating.

EXAMPLE 3.1.24 (Complex Pattern Recognition). Determine $\displaystyle\int \frac{x+1}{\sqrt{9-x^2}}\,dx$.

Solution. If the numerator of in the integrand were just x, we'd have an ordinary substitution problem (do you see why?). If the numerator were just 1 we'd have an arcsine problem (right?). Well, what we have is both. We can rewrite the integral as

[5] "Oh, look out!" She Came in Through the Bathroom Window—The Beatles.

the sum of two integrals, each of which we can solve as just described.

$$\begin{aligned}\int \frac{x+1}{\sqrt{9-x^2}}\,dx &= \int \frac{x}{\sqrt{9-x^2}}\,dx + \int \frac{1}{\sqrt{9-x^2}}\,dx \\ &= \int -\frac{\frac{1}{2}}{u^{1/2}}\,du + \int \frac{1}{\sqrt{a^2-x^2}}\,dx \\ &= -u^{1/2} + \arcsin\frac{x}{a} + c \\ &= -\sqrt{9-x^2} + \arcsin\frac{x}{3} + c\end{aligned}$$

For the first integral in the sum, let

$$u = 9 - x^2 \text{ so } du = -2x\,dx$$
$$\text{so } -\frac{1}{2}\,du = x\,dx.$$

and in the second $a^2 = 9$ or $a = 3$.

Check the answer by differentiating.

EXAMPLE 3.1.25 (Complex Pattern Recognition). Determine $\displaystyle\int \frac{\sec^2(4x)}{9+\tan^2(4x)}\,dx$.

Solution. If the denominator were just $9+\tan(4x)$ we could try a straightforward u-substitution and end up with a natural log, since the derivative of $9+\tan(4x)$ is $4\sec^2(4x)$ which we (almost) have in the numerator. But notice that we can think of $9+\tan^2(4x)$ as a^2+u^2, so what about an arctangent?

$$a^2 = 9 \Rightarrow a = 3 \qquad u^2 = \tan^2(4x) \Rightarrow u = \tan(4x)$$
$$du = 4\sec^2(4x)\,dx \text{ so } \tfrac{1}{4}du = \sec^2(4x)\,dx$$

Now substituting we find

$$\begin{aligned}\int \frac{\sec^2(4x)}{9+\tan^2(4x)}\,dx = \frac{1}{4}\int \frac{1}{a^2+u^2}\,dx &= \frac{1}{4}\cdot\frac{1}{a}\cdot\arctan\left(\frac{u}{a}\right) + c \\ &= \frac{1}{12}\arctan\left(\frac{\tan(4x)}{3}\right) + c\end{aligned}$$

Check the answer by differentiating.

YOU TRY IT 3.6. These integrals are similar looking and are done using substitution. However, the answers are quite different. Work each out.

$$(a)\ \int \frac{2t}{\sqrt{4-t^2}}\,dt \qquad (b)\ \int \frac{2t}{\sqrt{1-4t^4}}\,dt \qquad (c)\ \int \frac{t+4}{\sqrt{4-t^2}}\,dt$$

ANSWER TO YOU TRY IT 3.6.

(a) $-2\sqrt{4-t^2}+c$ (b) $\frac{1}{2}\arcsin 2t^2+c$ (c) $-\sqrt{4-t^2}+4\arcsin\frac{t}{2}+c$

YOU TRY IT 3.7. Now try this pair.

$$(a)\ \int \frac{e^x}{5+e^x}\,dx \qquad (b)\ \int \frac{e^x}{5+e^{2x}}\,dx$$

ANSWER TO YOU TRY IT 3.7.

(a) $\ln(5+e^x)+c$ (b) $\frac{1}{\sqrt{5}}\arctan\frac{e^x}{\sqrt{5}}+c$

One Last Type of Substitution

There are lots of tricky substitutions that can be discussed. Later in the term, we will use trig functions as a substitutions for x. That would seem to make things more complicated, but by that point in the term we will have developed more techniques to handle complicated trig integrals. For the moment we examine one last, less than obvious substitution.

EXAMPLE 3.1.26 (Unusual Substitution). Determine $\int \frac{x}{\sqrt{x-1}}\,dx$.

Solution. The usual substitution works, but it has a twist. As usual we let u be the inside function

$$u = x - 1 \text{ so } du = dx$$

But what do we do with the x in the numerator? Well,

$$u = x - 1 \text{ so } x = u + 1.$$

Carefully replace the pieces in the original integral.

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{1}{\sqrt{u}} \cdot (u-1)\,du = \int \frac{u-1}{u^{1/2}}\,du &= \int u^{1/2} - u^{-1/2}\,du \\ &= \frac{2}{3}u^{3/2} + 2u^{1/2} + c \\ &= \frac{2(x-1)^{3/2}}{3} + 2(x-1)^{1/2} + c.\end{aligned}$$

This is a bit messy, but you should be able to spot similar substitutions.

EXAMPLE 3.1.27 (Unusual Substitution). Determine $\int x\sqrt[3]{x+1}\,dx$.

Solution. This problem is very similar to the previous one. However, we will use a different type of substitution to solve it. This same method could also be used to solve the previous problem. Instead of letting $u = x + 1$ which is the inside function, let u be the entire root:

$$\begin{aligned} u = \sqrt[3]{x+1} \text{ so cubing gives } & u^3 = x + 1 \\ \text{so } & u^3 - 1 = x \\ \text{and } & 3u^2\,du = dx \end{aligned}$$

Wow! I like the way we get dx here! Now we can substitute. Carefully replace the pieces in the original integral.

$$\begin{aligned}\int x\sqrt[3]{x+1}\,dx = \int (u^3-1)\cdot u \cdot 3u^2\,du &= \int 3u^6 - 3u^3\,du \\ &= \tfrac{3}{7}u^7 - \tfrac{3}{4}u^4 + c = \tfrac{3}{7}(x+1)^{7/3} - \tfrac{3}{4}(+1)^{4/3} + c.\end{aligned}$$

Slick!

YOU TRY IT 3.8. Return to Example 3.1.26. Carry out a substitution in the style of Example 3.1.27 to solve the problem. You should get the same answer.

Change of Variables for Definite Integrals

When using u-substitution with definite integrals it is often convenient to determine the limits of integration for the variable u rather than having to convert the antiderivative back to a function of x. Here's the theorem that describes the process.

THEOREM 3.1.28 (Change of Variables for Definite Integrals). If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

A few examples will illustrate how this works.

EXAMPLE 3.1.29 (Substitution and Definite Integrals). Determine $\displaystyle\int_0^1 \frac{x^2}{2+3x^3}\,dx$.

Solution. Here

$$u = 2+3x^3 \text{ so } du = 9x^2\,dx$$
$$\text{so } \frac{1}{9}\,du = x^2\,dx.$$

Now change the limits using $u = 2+3x^3$.

$$\text{Lower limit: When } x = 0,\ u = 2+3(0)^3 = 2$$
$$\text{Upper limit: When } x = 1,\ u = 2+3(1)^3 = 5$$

Replace the pieces, including the limits, in the original integral.

$$\int_0^1 \frac{x^2}{2+3x^3}\,dx = \int_2^5 \frac{\frac{1}{9}}{u}\,du = \frac{1}{9}\ln|u|\Big|_2^5 = \frac{1}{9}[\ln 5 - \ln 2] = \frac{1}{9}\ln\frac{5}{2}.$$

We could also convert the answer back to the variable x, as we do with indefinite integrals, and then evaluate the integral using the original limits. If we did that here, we would get

$$\int \frac{x^2}{2+3x^3}\,dx = \frac{1}{9}\ln|u|\Big|_2^5 = \frac{1}{9}\ln|2+3x^3|\Big|_0^1 = \frac{1}{9}[\ln 5 - \ln 2] = \frac{1}{9}\ln\frac{5}{2},$$

which is the same answer we got earlier.

EXAMPLE 3.1.30 (Substitution and Definite Integrals). Determine $\displaystyle\int_0^\pi \sin 2x \cos 2x\,dx$.

Solution. Here we can let u be either $\sin 2x$ or $\cos 2x$. If we use the former,

$$u = \sin 2x \text{ so } du = 2\cos 2x\,dx$$
$$\text{so } \frac{1}{2}\,du = \cos 2x\,dx.$$

Now change the limits.

$$\text{Lower limit: When } x = 0,\ u = \sin 2(0) = 0$$
$$\text{Upper limit: When } x = \pi/2,\ u = \sin 2(\pi/2) = \sin\pi = 0$$

Since the upper and lower limits of integration are the same now, we know that the answer is 0.

$$\int_0^\pi \sin 2x\cos 2x\,dx = \int_0^0 \frac{1}{2}u\,du = 0.$$

EXAMPLE 3.1.31 (Substitution and Definite Integrals). Determine $\displaystyle\int_0^3 x\sqrt{9-x^2}\,dx$.

Solution. This is another simple substitution problem.

$$u = 9-x^2 \text{ so } du = -2x\,dx$$
$$\text{so } -\frac{1}{2}\,du = x\,dx.$$

Now change the limits using $u = 9 - x^2$.

$$\text{Lower limit: When } x = 0,\ u = 9 - (0)^2 = 9$$
$$\text{Upper limit: When } x = 3,\ u = 9 - (3)^2 = 0$$

Replace the pieces, including the limits, in the original integral. The new limits stay in the same order.

$$\int_0^3 x\sqrt{9-x^2}\,dx = \int_9^0 \frac{\frac{1}{2}}{u^{1/2}}\,du = \frac{1}{3}u^{3/2}\Big|_9^0 = \frac{1}{3}[0-27] = -9.$$

It is worth repeating that the new limits of integration remain in the same order as the original limits.

YOU TRY IT 3.9. Try these two problems that are a bit more theory oriented to see if you understand u-substitution conceptually.

(*a*) If $\int_1^4 f(x)\,dx = 5$, evaluate $\int_0^1 f(3x+1)\,dx$. Use u substitution.

(*b*) If $\int_0^4 f(x)\,dx = 1$, evaluate $\int_0^2 xf(x^2)\,dx$. Use u substitution.

ANSWER TO YOU TRY IT 3.9. (a) 5/3; (b) 1/2.

WEBWORK: Click to try Problems 64 through 70. Use GUEST login, if not in my course.

Two Trig Integrals

We begin with two key trig identities that you should **memorize** that will make your life and these integrals much simpler.

Two Key Identities.

$\sin^2 u = \frac{1}{2} - \frac{1}{2}\cos 2u$	(Half angle formula)
$\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$	

The half angle formulas are used to integrate $\sin^2 u$ or $\cos^2 u$ in the obvious way.

EXAMPLE 3.1.32. Determine $\displaystyle\int \cos^2(8x)\,dx$.

Solution. Use equation (1) above with $u = 8x$. Note the use of a 'mental adjustment.'

$$\int \cos^2(8x)\,dx = \int \tfrac{1}{2} + \tfrac{1}{2}\cos(16x)\,dx = \tfrac{1}{2}x + \tfrac{1}{32}\sin(16x) + c.$$

EXAMPLE 3.1.33. Determine $\displaystyle\int_0^{1/4} \sin^2(\pi x)\,dx$.

Solution. Use equation (1) above with $u = \pi x$. Note the use of a 'mental adjustment.'

$$\int_0^{1/4} \sin^2(\pi x)\,dx = \int_0^{1/4} \tfrac{1}{2} - \tfrac{1}{2}\cos(2\pi x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4\pi}\sin(2\pi x)\Big|_0^{1/4} = \tfrac{1}{8} - \tfrac{1}{4\pi}.$$

Multi-Substitutions

Sometimes you need to use more than one substitution. *Laissez les bons temps rouler!*

EXAMPLE 3.1.34. Determine $\int x^2 \cos^2(x^3)\sin(x^3)\,dx$. Start with the obvious substitution: $u = x^3$.

Solution. Let $u = x^3$ and $du = 3x^2\,dx$. We get

$$\int x^2 \cos^2(x^3)\sin(x^3)\,dx = \int \frac{1}{3}\cos^2 u \sin u\,du.$$

Now let $w = \cos u$ and $dw = -\sin u\,du$. Then

$$\int \frac{1}{3}\cos^2 u \sin u\,du = \int -\frac{1}{3}w^2\,dw = -\frac{1}{9}w^3 + c = -\frac{1}{9}\cos^3 u + c = -\frac{1}{9}\cos^3(x^3) + c.$$

EXAMPLE 3.1.35. This one is sweet: Gotta' love math: Determine $\int \frac{1}{\sqrt{1+\sqrt{1+x}}}\,dx$. Start with the substitution: $u = \sqrt{1+x}$.

Solution. If we let $u = \sqrt{1+x}$, then $du = \frac{1}{2\sqrt{1+x}}$ so $2\sqrt{1+x}\,du = dx$ So $2u\,du = dx$. Whoa! Now

$$\begin{aligned}\int \frac{1}{\sqrt{1+\sqrt{1+x}}}\,dx &= \int \frac{2u}{\sqrt{1+u}} \\ &\overset{\text{Like Example 3.1.26}}{=} 2\left[\frac{2}{3}(u+1)^{3/2} + 2(u+1)^{1/2}\right] + c \\ &= \frac{4}{3}(1+\sqrt{1+x})^{3/2} - 4(1+\sqrt{1+x})^{1/2} + c.\end{aligned}$$

YOU TRY IT 3.10 (Integral Mixer). Evaluate these definite integrals. A number of techniques are required. Switch the limits of integration when appropriate.

(a) $\int_0^3 \frac{1}{\sqrt{4-x}}\,dx$ (b) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ (c) $\int_0^{\sqrt{3}} \frac{x}{9+x^4}\,dx$

(d) $\int_0^1 \frac{x^9}{1+x^{20}}\,dx$ (e) $\int_0^2 \frac{2x}{9+x^2}\,dx$ (f) $\int \frac{\arctan 2x}{1+4x^2}\,dx$ (g) $\int_0^3 \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx$

(a) 2 (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{24}$ (d) $\frac{\pi}{40}$ (e) $\ln\frac{13}{9}$ (f) $\frac{(\arctan 2x)^2}{4} + c$ (g) $\ln 2$

ANSWER TO YOU TRY IT 3.10.