Application: Volume

6.1 Overture

In this chapter we present another application of the definite integral, this time to find volumes of certain solids. As important as this particular application is, more important is to recognize a pattern or theme that will allow us to apply the notion of a definite integral to other contexts. For continuous functions we know that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx.$$

Notice that the Riemann sum is actually a sum of products. Many quantities can be expressed as a sum of such products—products where the entire quantity has been divided into smaller 'approximating' pieces. (Think of the rectangles that approximate the thin strips of area under a curve which has been subdivided by a regular partition. The area of such approximating rectangles is a product: $b \times h$.) Whenever we can approximate by using a sum of products in this way, we can compute the entire quantity not as a sum, but as a definite integral. We explore this powerful idea—which I call 'subdivide and conquer'—below.

Note: The development of this material is slightly different than in your text, though the same results are achieved.

Cylinders

In high school geometry one is introduced to shapes known as cylinders. Typically we think of a cylinder as the shape of a soup can. (See Figure 6.1.) A can has a circular base a which is moved along an axis perpendicular to the base to create the cylinder. Where the base stops moving, the top of the can is formed.

The volume of a cylinder is determined by the base and the length of the axis perpendicular to the base. More precisely

$$\text{Volume of a Cylinder} = \text{area of the base} \times \text{height}.$$

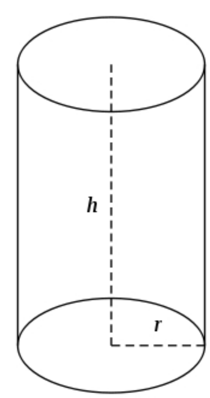


Figure 6.1: A circular cylinder is determined by its circular base and a perpendicular axis. (Diagram from `wikipedia.org/wiki/Cylinder_(geometry)`).

Notice that this is a product.

Mathematicians treat the notion of a cylinder more generally by allowing the base to be any finite plane region. Take any plane region B and move it a fixed distance h along an axis perpendicular to the base B. The resulting solid that is 'swept out' by this motion is a **cylinder**. See Figure 6.2.

The volume of any cylinder is still the product

$$\boxed{\text{Volume of a Cylinder} = \text{area of the base} \times \text{height.}} \tag{6.1}$$

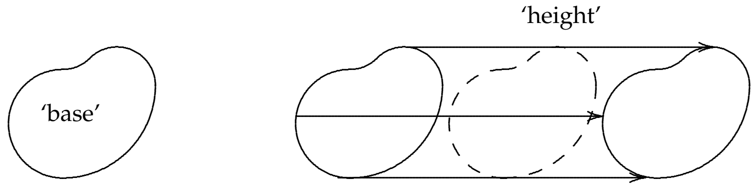


Figure 6.2: Left: A plane region. Right: Moving the plane region along an axis perpendicular to the region produces a **cylinder**. The dashed figure represents a cross-sectional slice perpendicular to the axis.

Notice that a cardboard carton also satisfies this more general notion of a cylinder. Its base is the bottom of the carton which we can think of moving vertically a distance equal to the height of the box to form a 'rectangular' cylinder. The volume of the carton is (area of the base) $\times$ height, where the base is a rectangle. The rectangle area is $l \times w$, so the volume of the box is is the familiar formula

$$\text{area of the base} \times \text{height} = l \times w \times h.$$

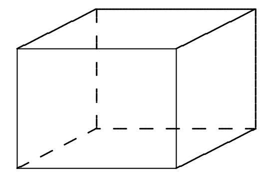


Figure 6.3: A rectangular box can be thought of as a cylinder determined by its rectangular base and a perpendicular axis.

Obviously computing the volumes of cylinders (including boxes) is easy using the formula in (6.1). How do we use this formula in more general settings to obtain an integral?

A Loaf of Bread

Consider a nice crusty loaf of artisan bread. How might we determine its volume? Let's place the loaf on an axis—suppose the loaf lies between a and b as shown in Figure 6.4.

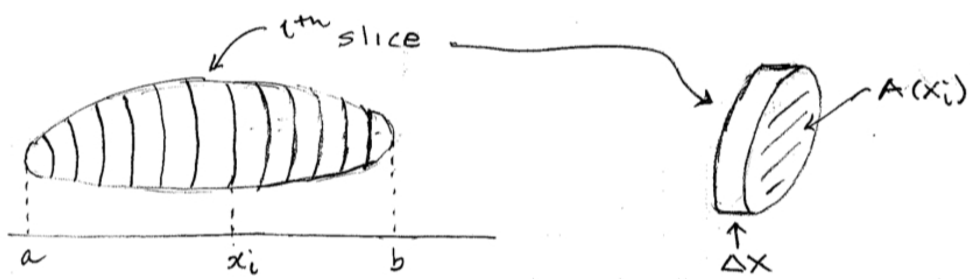


Figure 6.4: Left: A loaf of bread cut into n slices. Right: The ith slice is almost a cylinder.

Slice the loaf into n equal slices, each of width Δx. Let V_i denote the volume of the ith slice. Then the volume of the loaf is the sum of the volume of all the slices ('subdivide and conquer').

$$\text{Volume of Loaf} = \sum_{i=1}^{n} \text{Volume of Slice } i = \sum_{i=1}^{n} V_i.$$

How do we determine the volume of the a slice? When we extract the ith slice from the loaf (see Figure 6.4), we see that it is almost the shape of a cylinder—its two faces or cross-sections are nearly identical. Since we nearly have a cylinder,

$$V_i \approx (\text{area of the base}) \times \text{height}.$$

But the area of the base is just the cross-sectional area of the slice and the 'height' is really the width Δx of the slice, so

$$V_i \approx (\text{area of cross-section of } i\text{th slice}) \times \Delta x.$$

If we let $A(x_i)$ denote the cross-sectional area of the ith slice ((see Figure 6.4), then

$$V_i \approx A(x_i)\Delta x.$$

So the volume of the entire loaf is approximated by

$$\text{Volume of Loaf} = \sum_{i=1}^{n} V_i \approx \sum_{i=1}^{n} A(x_i)\Delta x.$$

The approximation is improved by letting the number of slices get large and then taking the limit. In other words,

$$\text{Volume of Loaf} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx,$$

where we have used the fact that if the cross-sectional area is a continuous function, then the limit of the Riemann sums exists and is a definite integral. More precisely, we have proved

THEOREM 6.1.1 (Volume Formula). Let V be the volume of a solid that lies between $x = a$ and $x = b$. If for each x in the interval $[a, b]$ the cross-sectional area perpendicular to the x-axis is given by the continuous function $A(x)$, then the **volume** V is the solid is

$$V = \int_a^b A(x)\,dx.$$

Note: If the slices are taken perpendicular to the y-axis on the interval $[c, d]$ and the cross-sectional area is $A(y)$, then

$$V = \int_c^d A(y)\,dy.$$

Stop! Notice that we used the 'subdivide and conquer' process to approximate the quantity we wish to determine. That is, we subdivided the volume slicing it into 'approximating cylinders' whose volume we know how to compute. We refined this approximation by letting the number of slices get large. Taking the limit of this process answered our question. Identifying that limit with an integral makes it possible to easily (!) compute the volume in question. OK, time for some examples.

Examples

EXAMPLE 6.1.2. A crystal prism is 20 cm long (figure on the left below). Its cross-sections are right triangles whose heights are formed from the line $y = \frac{1}{2}x$ and whose bases are twice the height. Find the volume of the prism.

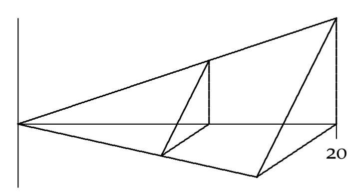


Figure 6.5: A prism with a representative right triangle cross-section.

Solution. The cross-sections are right triangles whose heights are $\frac{1}{2}x$ and the base is twice the height. So the cross-sectional area is

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{2}x\right)x = \frac{1}{4}x^2.$$

Using Theorem 6.1.1 the volume of the prism is

$$V = \int_a^b A(x)\,dx = \int_0^{20} \frac{1}{4}x^2\,dx = \frac{1}{12}x^3\Big|_0^{20} = \frac{2000}{3} \text{ cc}.$$

EXAMPLE 6.1.3. Find the volume of the *Great Pyramid of Cheops* which has a square base 750 ft on each edge and a height of 500 ft.

Solution. In order to simplify the mathematics, it will be useful to draw the pyramid upside-down (see Figure 6.6).

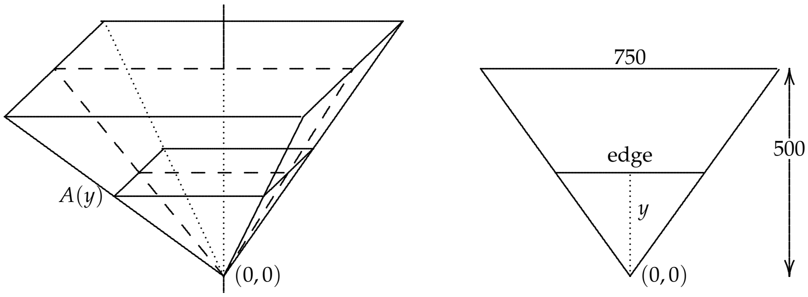


Figure 6.6: Left: The Great Pyramid of Cheops upside down. Cross-sections perpendicular to the y-axis are squares. Right: The relation between the height of the cross-section and its edge length.

The cross-sections are squares with area $A(y)$. We need to determine the area of the square at height y, so we need to find the length of the edge of the square at height y. To do this we can use similar triangles, see the right half of Figure 6.6. We have

$$\frac{\text{edge}}{y} = \frac{750}{500} \Rightarrow \text{edge} = \frac{3y}{2}.$$

Since the cross-section is a square,

$$A(y) = (\text{edge})^2 = \left(\frac{3y}{2}\right)^2 = \frac{9y^2}{4}.$$

Therefore, by Theorem 6.1.1

$$V = \int_c^d A(y)\,dy = \int_0^{500} \frac{9y^2}{4}\,dy = \frac{3}{4}y^3\Big|_0^{500} = 93,750,000 \text{ cu ft.}$$

YOU TRY IT 6.1. When I was in Tasmania in 1998, I bought a beautiful wedge-shaped wooden doorstop. It is 15 cm long and 5 cm high at its tall end and 4 cm wide. See Figure 6.7 below. Find the volume of this wedge using calculus. Hint: Find the equation of the line that forms one of the top edges. (Why should the answer be 150 cu. cm.?)

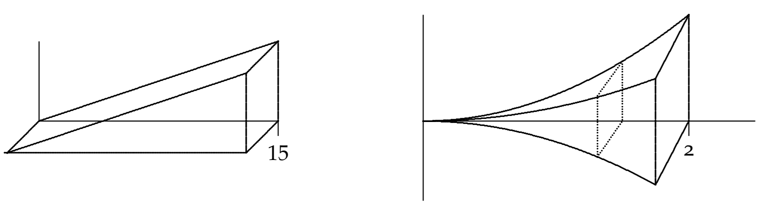


Figure 6.7: Left: The wedge doorstop for YOU TRY IT 6.1 . Right: The prism for YOU TRY IT 6.2 .

YOU TRY IT 6.2. A crystal prism is 2 cm long. Its cross-sections are **squares** with heights formed by the curve $y = x^2$. See Figure 6.7 above. Find the volume of the prism.

YOU TRY IT 6.3. Use Theorem 6.1.1 to prove that the volume formula for a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$. Hint: Draw the cone vertically with its vertex at the origin. Determine the equation of the line that forms the 'right-hand' edge of the cone. Use that linear equation to determine the radius of the circular cross-sections of the cone.

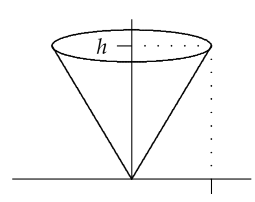
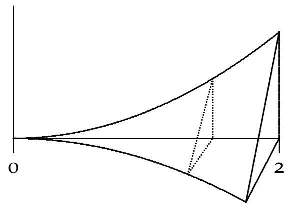


Figure 6.8: Determine the volume of a cone of radius r and height h.

YOU TRY IT 6.4. A crystal prism is 2 cm long. Its cross-sections are isosceles right triangles. The heights are formed by the curve $y = x^2$. See Figure 6.9 below. Find the volume of the prism.



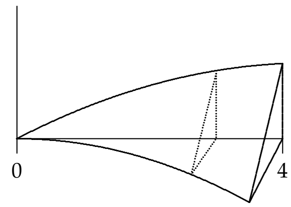


Figure 6.9: Left: The prism for YOU TRY IT 6.4 . Right: The prism for YOU TRY IT 6.5 .

YOU TRY IT 6.5. A crystal prism is 4 cm long. Its cross-sections are right triangles. The heights are formed by the curve $y = 2\sqrt{x}$ and the bases by the curve $y = x^2$. See Figure 6.9 above. Find the volume of the prism.

YOU TRY IT 6.6 (The Great Pyramid of Geneva). The pyramid at the Pyramid Mall in Geneva at the crest of Bean's Hill has a square base with edges that measure 300 meters. Its height is 150 meters. Find its volume. Hint: Review the Pyramid of Cheops problem. The equations are simpler if you turn the pryamid upside down and use cross-sections perpendicular to the y-axis. (Answer: 4,500,000 cu. m.)

YOU TRY IT 6.7. A field biologist is doing a survey of a small wooded forest. She is interested in finding the volume of tree trunks from the forest floor to a point 2 meters above the ground. Since she cannot measure the volume directly, she uses a pair of tree calipers to measure the radius of the tree at 40 cm intervals over the range from 0 to 200 centimeters. She brings the data to you (see table below) and asks you to provide a reliable estimate on the volume of the tree trunk in cubic centimeters. How can you do so using Riemann sums? What estimate should you use to get a reasonably good approximation? Explain your reasoning.

Height (h)	0	40	80	120	160	200
Radius (r)	28	30	26	24	20	18

YOU TRY IT 6.8 (Theory). Suppose we form a regular partition of the interval $[a, b]$ and create the Riemann sum:

$$S_n = \sum_{k=1}^{n} \sqrt{1 + [f'(x)]^2} \Delta x,$$

where $f'(x)$ is a continuous function. Express $\lim_{n\to\infty} S_n$ as an integral. We will use this in class in a couple of days.

6.2 *Volumes of Revolution: The Disk Method*

One of the simplest applications of integration (Theorem 6.1.1)—and the accumulation process—is to determine so-called volumes of revolution. In this section we will concentrate on a method known as the **disk method**.

Solids of Revolution

If a region in the plane is revolved about a line in the same plane, the resulting object is a **solid of revolution**, and the line is called the **axis of revolution**. The following situation is typical of the problems we will encounter.

Solids of Revolution from Areas Under Curves. Suppose that $y = f(x)$ is a continuous (non-negative) function on the interval $[a, b]$. Rotate the region under the f between $x = a$ and $x = b$ around the the x-axis and determine the volume of the resulting solid of revolution. See Figure 6.10

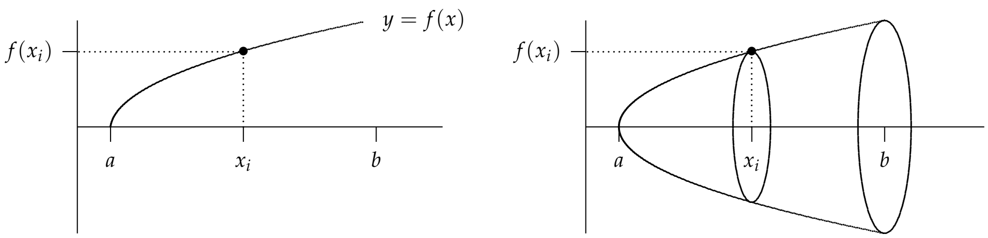


Figure 6.10: Left: The region under the continuous curve $y = f(x)$ on the interval $[a, b]$. Right: The solid generated by rotating the region about the x-axis. Note: The point $(x_i, f(x_i))$ on the curve traces out a circular cross-section of radius $r = f(x_i)$ when rotated.

Once we know the cross-sectional areas of the solid, we can use Theorem 6.1.1 to determine the volume. But as Figure 6.10 shows, when the point $(x_i, f(x_i))$ on the curve is rotated about the x-axis, it forms a circular cross-section of radius $R = f(x_i)$. Therefore, the cross-sectional area at x_i is

$$A(x_i) = \pi R^2 = \pi[f(x_i)]^2.$$

Since f is continuous, so is $\pi[f(x)]^2$ and consequently Theorem 6.1.1 applies.

$$\text{Volume of Solid of Revolution} = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx.$$

Of course, we could use this same process if we rotated the region about the y-axis and integrated along the y-axis. We gather these results together and state them as a theorem.

THEOREM 6.2.1 (The Disk Method). If V is the volume of the solid of revolution determined by rotating the continuous function $f(x)$ on the interval $[a, b]$ about the x-axis, then

$$V = \pi \int_a^b [f(x)]^2\,dx. \tag{6.2}$$

If V is the volume of the solid of revolution determined by rotating the continuous function $f(y)$ on the interval $[c, d]$ about the y-axis, then

$$V = \pi \int_c^d [f(y)]^2\,dy. \tag{6.3}$$

Another Development of the Disk Method Using Riemann Sums

Instead of using Theorem 6.1.1, we could obtain Theorem 6.2.1 directly by using the 'subdivide and conquer' strategy once again. Since we will use this strategy in later situations, let's quickly go through the argument here.

There are often several ways to prove a result in mathematics. I hope one of these two will resonate with you.

As above, we start with a continuous function on $[a, b]$. This time, though, we create a regular partition of $[a, b]$ using n intervals and draw the corresponding approximating rectangles of equal width Δx. In left half of Figure 6.11 we have drawn a single representative approximating rectangle on the ith subinterval.

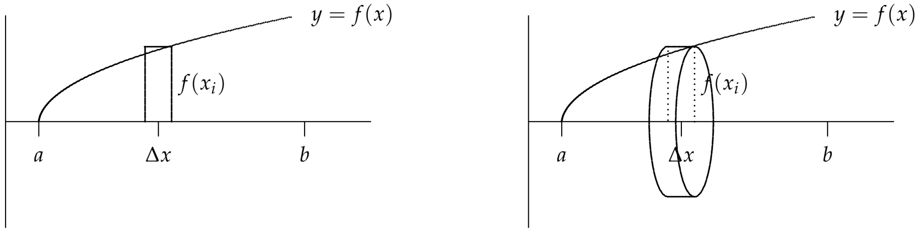


Figure 6.11: Left: The region under the continuous curve $y = f(x)$ on the interval $[a, b]$ and a representative rectangle. Right: The disk (cylinder) of radius $R = f(x_i)$ generated by rotating the representative rectangle about the x-axis. The volume or this disk is $\pi R^2 w = \pi[f(x_i)]^2 \Delta x$.

Rotating each representative rectangle creates a **representative disk** (cylinder) of radius $R = f(x_i)$. (See the right half of Figure 6.11.) The volume of this cylinder is given by (6.1)

$$\text{Volume of a Cylinder} = (\text{area of the base}) \times \text{height}.$$

In this case when the disk is situated on its side, we think of the height as the 'width' Δx of the disk. Moreover, since the base is a circle, its area is $\pi R^2 = \pi[f(x_i)]^2$ so

$$\text{Volume of a representative disk} = \Delta V_i = \pi[f(x_i)]^2 \Delta x.$$

To determine the volume of entire solid of revolution, we take each approximating rectangle, form the corresponding disks (see the middle panel of Figure 6.12) and sum the resulting volumes, it generates a **representative disk** whose volume is

$$\Delta V = \pi R^2 \Delta x = \pi[R(x_i)]^2 \Delta x.$$

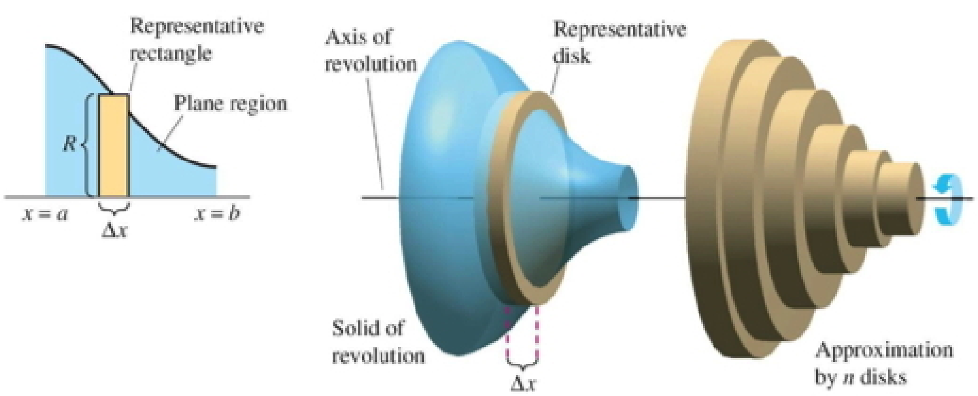


Figure 6.12: A general solid of revolution and its approximation by a series of n disks. (Diagram from Larson & Edwards)

Approximating the volume of the entire solid by n such disks (see the right-hand panel of Figure 6.12) of width Δx and radius $f(x_i)$ produces a Riemann sum

$$\text{Volume of Revolution} \approx \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \pi \sum_{i=1}^{n} [f(x_i)]^2 \Delta x. \quad (6.4)$$

As usual, to improve the approximation we let the number of subdivisions $n \to \infty$ and take a limit. Recall from our earlier work with Riemann sums, this limit exists because $[f(x)]^2$ is continuous on $[a,b]$ since $f(x)$ is continuous there.

$$\text{Volume of Revolution} = \lim_{n\to\infty} \pi \sum_{i=1}^{n} [f(x_i)]^2 \Delta x = \pi \int_a^b [f(x)]^2\, dx \qquad (6.5)$$

where we have used the fact that the limit of a Riemann sum is a definite integral. This is the same result we obtained in Theorem 6.2.1. Ee could use this same process if we rotated the region about the y-axis and integrated along the y-axis.

Stop! Notice how we used the 'subdivide and conquer' process to approximate the quantity we wish to determine. That is we have subdivided the volume into 'approximating disks' whose volume we know how to compute. We have then refined this approximation by using finer and finer subdivisions. Taking the limit of this process provides the answer to our question. Identifying that limit with an integral makes it possible to easily (!) compute the volume in question. OK, time for some examples.

I'll admit it is hard to draw figures like Figure 6.12. However, drawing a representative rectangle for the region in question, as in the left half of Figure 6.11 is usually sufficient to set up the required volume integral.

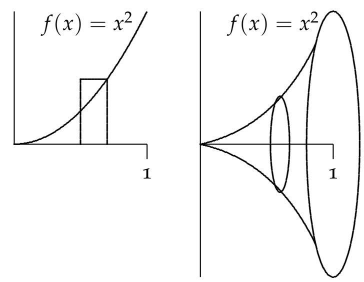


Figure 6.13: Left: A representative rectangle for the curve $y = x^2$. Right: A representative circular slice for the curve $y = x^2$ rotated about the x-axis.

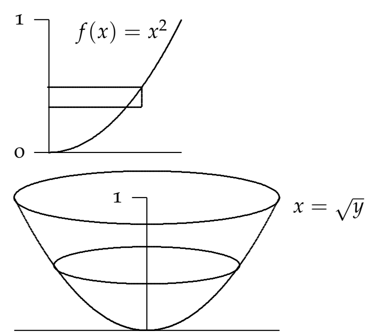


Figure 6.14: Left: A representative rectangle for the region between the curve $y = x^2$ ($x = \sqrt{y}$) and the y-axis. Right: A representative circular slice for the curve $x = \sqrt{y}$ rotated about the x-axis.

Examples

Let's start with a couple of easy ones.

EXAMPLE 6.2.2. Let $y = f(x) = x^2$ on the interval $[0,1]$. Rotate the region between the curve and the x-axis around the x-axis and find the volume of the resulting solid.

Solution. Using Theorem 6.2.1

$$V = \pi \int_a^b [f(x)]^2\, dx = \pi \int_0^1 [x^2]^2\, dx = \frac{\pi x^5}{5}\bigg|_0^1 = \frac{\pi}{5}.$$

Wow, that was easy!

EXAMPLE 6.2.3. Let $y = f(x) = x^2$ on the interval $[0,1]$. Rotate the region between the curve and the y-axis *around the y-axis* and find the volume of the resulting solid.

Solution. The region is not the same one as in Example 6.2.2. It lies between the y-axis and the curve, not the x-axis. See Figure 6.14.

Since the rotation is about the y-axis, we need to solve for x as a function of y. Since $y = x^2$, then $x = \sqrt{y}$. Notice that the region lies over the interval $[0,1]$ on the y-axis now. Using Theorem 6.2.1

$$V = \pi \int_c^d [g(y)]^2\, dy = \pi \int_0^1 [\sqrt{y}]^2\, dy = \frac{\pi y^2}{2}\bigg|_0^1 = \frac{\pi}{2}.$$

EXAMPLE 6.2.4. Find the volume of a sphere of radius r which can be obtained by rotating the semi-circle $f(x) = \sqrt{r^2 - x^2}$ about the x-axis.

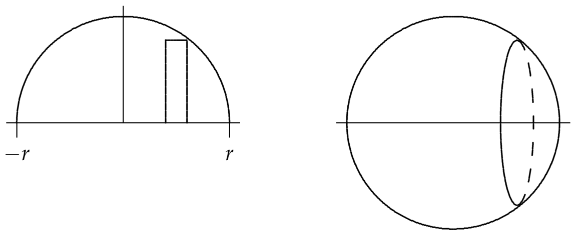


Figure 6.15: Left: A representative rectangle for the curve $y = \sqrt{r^2 - x^2}$. Right: A representative circular slice for the sphere that results when rotating the semi-circle about the x-axis.

Solution. Using Theorem 6.2.1

$$\begin{aligned} V = \pi \int_a^b [f(x)]^2\,dx &= \pi \int_{-r}^{r} [\sqrt{r^2 - x^2}]^2\,dx \\ &= \pi \int_{-r}^{r} r^2 - x^2\,dx \\ &= \pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^{r} \\ &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

Amazing! We have derived the volume formula of a sphere from the volume by disks formula.