6.2 *Volumes of Revolution: The Disk Method*

One of the simplest applications of integration (Theorem 6.1.1)—and the accumulation process—is to determine so-called volumes of revolution. In this section we will concentrate on a method known as the **disk method**.

Solids of Revolution

If a region in the plane is revolved about a line in the same plane, the resulting object is a **solid of revolution**, and the line is called the **axis of revolution**. The following situation is typical of the problems we will encounter.

Solids of Revolution from Areas Under Curves. Suppose that $y = f(x)$ is a continuous (non-negative) function on the interval $[a, b]$. Rotate the region under the f between $x = a$ and $x = b$ around the the x-axis and determine the volume of the resulting solid of revolution. See Figure 6.10

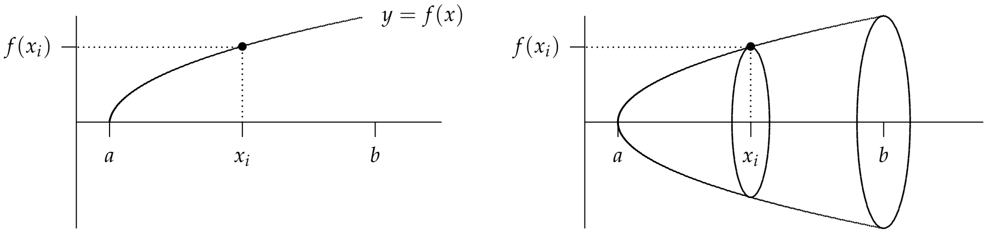

Figure 6.10: Left: The region under the continuous curve $y = f(x)$ on the interval $[a, b]$. Right: The solid generated by rotating the region about the x-axis. Note: The point $(x_i, f(x_i))$ on the curve traces out a circular cross-section of radius $r = f(x_i)$ when rotated.

Once we know the cross-sectional areas of the solid, we can use Theorem 6.1.1 to determine the volume. But as Figure 6.10 shows, when the point $(x_i, f(x_i))$ on the curve is rotated about the x-axis, it forms a circular cross-section of radius $R = f(x_i)$. Therefore, the cross-sectional area at x_i is

$$A(x_i) = \pi R^2 = \pi[f(x_i)]^2.$$

Since f is continuous, so is $\pi[f(x)]^2$ and consequently Theorem 6.1.1 applies.

$$\text{Volume of Solid of Revolution} = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx.$$

Of course, we could use this same process if we rotated the region about the y-axis and integrated along the y-axis. We gather these results together and state them as a theorem.

THEOREM 6.2.1 (The Disk Method). If V is the volume of the solid of revolution determined by rotating the continuous function $f(x)$ on the interval $[a, b]$ about the x-axis, then

$$V = \pi \int_a^b [f(x)]^2\,dx. \qquad (6.2)$$

If V is the volume of the solid of revolution determined by rotating the continuous function $f(y)$ on the interval $[c, d]$ about the y-axis, then

$$V = \pi \int_c^d [f(y)]^2\,dy. \qquad (6.3)$$

Another Development of the Disk Method Using Riemann Sums

Instead of using Theorem 6.1.1, we could obtain Theorem 6.2.1 directly by using the 'subdivide and conquer' strategy once again. Since we will use this strategy in later situations, let's quickly go through the argument here.

There are often several ways to prove a result in mathematics. I hope one of these two will resonate with you.

As above, we start with a continuous function on $[a, b]$. This time, though, we create a regular partition of $[a, b]$ using n intervals and draw the corresponding approximating rectangles of equal width Δx. In left half of Figure 6.11 we have drawn a single representative approximating rectangle on the ith subinterval.

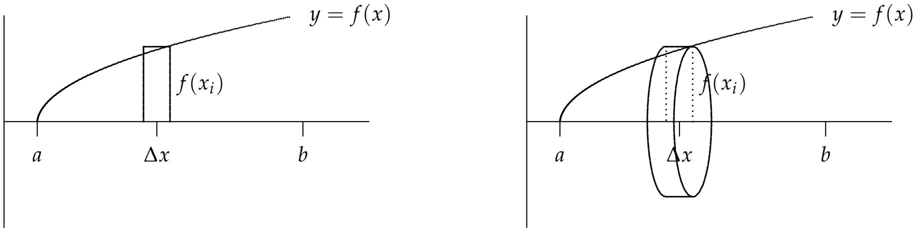

Figure 6.11: Left: The region under the continuous curve $y = f(x)$ on the interval $[a, b]$ and a representative rectangle. Right: The disk (cylinder) of radius $R = f(x_i)$ generated by rotating the representative rectangle about the x-axis. The volume or this disk is $\pi R^2 w = \pi[f(x_i)]^2 \Delta x$.

Rotating each representative rectangle creates a **representative disk** (cylinder) of radius $R = f(x_i)$. (See the right half of Figure 6.11.) The volume of this cylinder is given by (6.1)

$$\text{Volume of a Cylinder} = (\text{area of the base}) \times \text{height}.$$

In this case when the disk is situated on its side, we think of the height as the 'width' Δx of the disk. Moreover, since the base is a circle, its area is $\pi R^2 = \pi[f(x_i)]^2$ so

$$\text{Volume of a representative disk} = \Delta V_i = \pi[f(x_i)]^2 \Delta x.$$

To determine the volume of entire solid of revolution, we take each approximating rectangle, form the corresponding disks (see the middle panel of Figure 6.12) and sum the resulting volumes, it generates a **representative disk** whose volume is

$$\Delta V = \pi R^2 \Delta x = \pi[R(x_i)]^2 \Delta x.$$

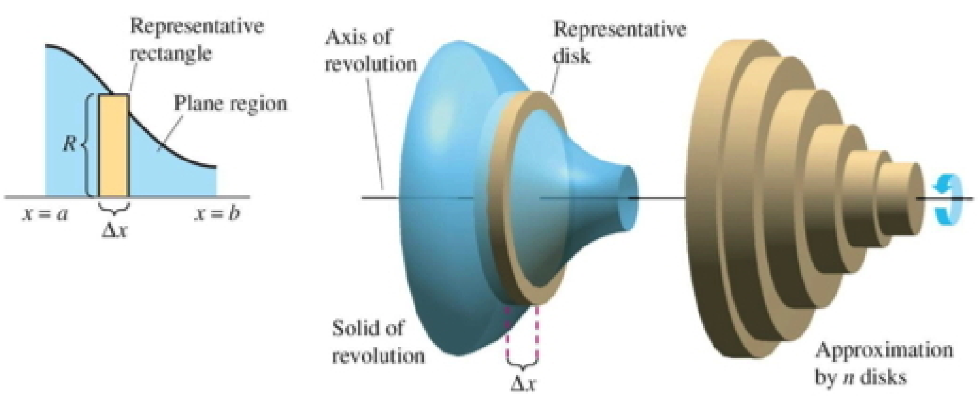

Figure 6.12: A general solid of revolution and its approximation by a series of n disks. (Diagram from Larson & Edwards)

Approximating the volume of the entire solid by n such disks (see the right-hand panel of Figure 6.12) of width Δx and radius $f(x_i)$ produces a Riemann sum

$$\text{Volume of Revolution} \approx \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \pi \sum_{i=1}^{n} [f(x_i)]^2 \Delta x. \qquad (6.4)$$

As usual, to improve the approximation we let the number of subdivisions $n \to \infty$ and take a limit. Recall from our earlier work with Riemann sums, this limit exists because $[f(x)]^2$ is continuous on $[a, b]$ since $f(x)$ is continuous there.

$$\text{Volume of Revolution} = \lim_{n\to\infty} \pi \sum_{i=1}^{n} [f(x_i)]^2 \Delta x = \pi \int_a^b [f(x)]^2 \, dx \tag{6.5}$$

where we have used the fact that the limit of a Riemann sum is a definite integral. This is the same result we obtained in Theorem 6.2.1. Ee could use this same process if we rotated the region about the y-axis and integrated along the y-axis.

Stop! Notice how we used the 'subdivide and conquer' process to approximate the quantity we wish to determine. That is we have subdivided the volume into 'approximating disks' whose volume we know how to compute. We have then refined this approximation by using finer and finer subdivisions. Taking the limit of this process provides the answer to our question. Identifying that limit with an integral makes it possible to easily (!) compute the volume in question. OK, time for some examples.

I'll admit it is hard to draw figures like Figure 6.12. However, drawing a representative rectangle for the region in question, as in the left half of Figure 6.11 is usually sufficient to set up the required volume integral.

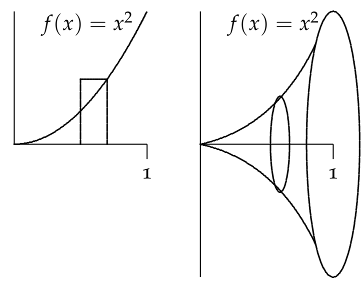

Figure 6.13: Left: A representative rectangle for the curve $y = x^2$. Right: A representative circular slice for the curve $y = x^2$ rotated about the x-axis.

Examples

Let's start with a couple of easy ones.

EXAMPLE 6.2.2. Let $y = f(x) = x^2$ on the interval $[0, 1]$. Rotate the region between the curve and the x-axis around the x-axis and find the volume of the resulting solid.

Solution. Using Theorem 6.2.1

$$V = \pi \int_a^b [f(x)]^2 \, dx = \pi \int_0^1 [x^2]^2 \, dx = \frac{\pi x^5}{5} \bigg|_0^1 = \frac{\pi}{5}.$$

Wow, that was easy!

EXAMPLE 6.2.3. Let $y = f(x) = x^2$ on the interval $[0, 1]$. Rotate the region between the curve and the y-axis *around the y-axis* and find the volume of the resulting solid.

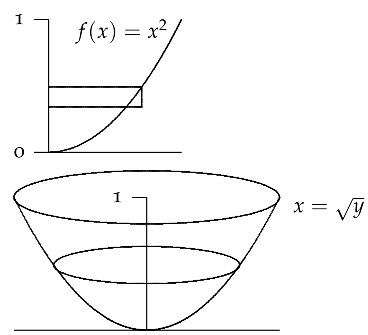

Figure 6.14: Left: A representative rectangle for the region between the curve $y = x^2$ ($x = \sqrt{y}$) and the y-axis. Right: A representative circular slice for the curve $x = \sqrt{y}$ rotated about the x-axis.

Solution. The region is not the same one as in Example 6.2.2. It lies between the y-axis and the curve, not the x-axis. See Figure 6.14.

Since the rotation is about the y-axis, we need to solve for x as a function of y. Since $y = x^2$, then $x = \sqrt{y}$. Notice that the region lies over the interval $[0, 1]$ on the y-axis now. Using Theorem 6.2.1

$$V = \pi \int_c^d [g(y)]^2 \, dy = \pi \int_0^1 [\sqrt{y}]^2 \, dy = \frac{\pi y^2}{2} \bigg|_0^1 = \frac{\pi}{2}.$$

EXAMPLE 6.2.4. Find the volume of a sphere of radius r which can be obtained by rotating the semi-circle $f(x) = \sqrt{r^2 - x^2}$ about the x-axis.

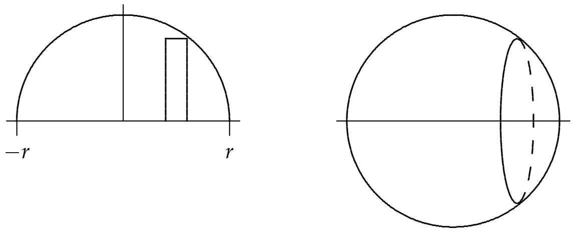

Figure 6.15: Left: A representative rectangle for the curve $y = \sqrt{r^2 - x^2}$. Right: A representative circular slice for the sphere that results when rotating the semi-circle about the x-axis.

Solution. Using Theorem 6.2.1

$$\begin{aligned} V = \pi \int_a^b [f(x)]^2\,dx &= \pi \int_{-r}^{r} [\sqrt{r^2 - x^2}]^2\,dx \\ &= \pi \int_{-r}^{r} r^2 - x^2\,dx \\ &= \pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^{r} \\ &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

Amazing! We have derived the volume formula of a sphere from the volume by disks formula.

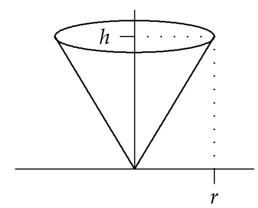

Figure 6.16: Determine the volume of a cone of radius r and height h.

YOU TRY IT 6.9. Find the volume of a cone of radius r and height h by rotating the line through $(0, 0)$ and (r, h) about the y-axis. (See Figure 6.16.)

EXAMPLE 6.2.5 (Two Pieces). Consider the region enclosed by the curves $y = \sqrt{x}$, $y = 6 - x$, and the x-axis. Rotate this region about the x-axis and find the resulting volume.

Solution. It is important to sketch the region to see the relationship between the curves. Pay particular attention to the bounding curves. Draw representative rectangles. This will help you set up the appropriate integrals. It is less important to try to draw a very accurate three-dimensional picture.

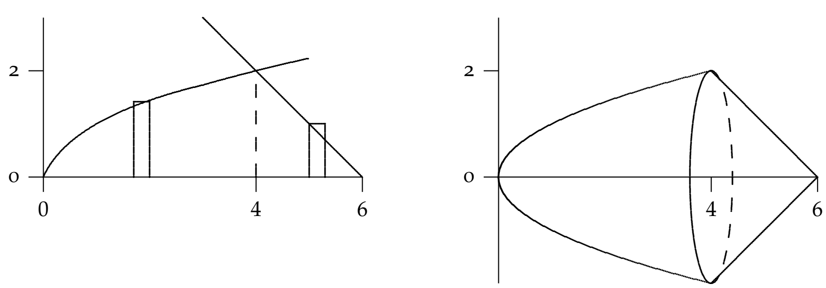

Figure 6.17: Left: The region enclosed by the curves $y = \sqrt{x}$, $y = 6 - x$, and the x-axis and two representative rectangles. Right: The resulting solid of revolution about the x-axis is formed by two distinct pieces each requiring its own integral.

First determine where the curves intersect: Obviously $y = \sqrt{x}$ meets the x-axis at $x = 0$ and $y = 6 - x$ meets the x-axis at $x = 6$. For $y = 6 - x$ and $y = \sqrt{x}$,

$$\begin{aligned} 6 - x = \sqrt{x} \Rightarrow 36 - 12x + x^2 = x \Rightarrow x^2 - 13x + 36 = (x-4)(x+9) = 0 \\ \Rightarrow x = 4,\ (x = -9 \text{ is not in the domain}). \end{aligned}$$

From Figure 6.17 we see that the solid is made up of two separate pieces (the top curve changes at $x = 4$) and each requires its own integral. Using Theorem 6.2.1

$$\begin{aligned} V &= \pi \int_0^4 [\sqrt{x}]^2\,dx + \pi \int_4^6 [6-x]^2\,dx \\ &= \pi \int_0^4 x\,dx + \pi \int_4^6 [6-x]^2\,dx \\ &= \frac{\pi x^2}{2}\bigg|_0^4 + \frac{-\pi(6-x)^3}{3}\bigg|_4^6 \\ &= (8\pi - 0) + \left(0 - \frac{(-8\pi)}{3}\right) \\ &= \frac{32\pi}{3}. \end{aligned}$$

Note: We used a 'mental adjustment' to do the second integral (with $u = 6 - x$ and $du = -dx$). Overall, the integration is easy once the problem is set up correctly. Be sure you have the correct region.

EXAMPLE 6.2.6 (One Piece: Two integrals)**.** Reconsider the same region as in Example 6.2.5 enclosed by the curves $y = \sqrt{x}$, $y = 6 - x$, and the x-axis. Now rotate this region about the y-axis instead and find the resulting volume.

Solution. OK, the region is the same as above. Here is where you have to be very careful. Since the rotation is about the y-axis, the strips are horizontal this time. Notice that there are two strips. When this region is rotated about the y-axis, the solid will have a 'hollowed out' center portion (see Figure 6.18). Thus, we must take the outer region formed by the curve $y = 6 - x$ and *subtract* the inner region formed by $y = \sqrt{x}$ from it.

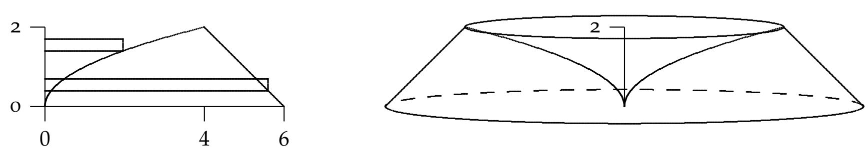

Figure 6.18: Left: The region enclosed by the curves $y = \sqrt{x}$, $y = 6 - x$, and the x-axis and two representative rectangles. Right: The resulting solid of revolution about the x-axis (a 'volcano') is formed by two distinct pieces each requiring its own integral.

Since the rotation is about the y-axis, to use the disk method we need to write the curves in the form x as a function of y. We have

$$y = \sqrt{x} \Rightarrow x = y^2 \text{ and } y = 6 - x \Rightarrow x = 6 - y.$$

The outer curve is $x = 6 - y$ and the inner curve is $x = y^2$ from the perspective of the y-axis. From the previous problem we already know the intersection points. However we need their y-coordinates. When $x = 4$ the corresponding y-coordinate (using either $y = \sqrt{x}$ or $y = 6 - x$) is $y = 2$. The other coordinate is $y = 0$. So this time using Theorem 6.2.1 (note the subtraction of the inner volume)

$$\begin{aligned} V &= \text{Outer Volume} - \text{Inner Volume} \\ &= \pi \int_0^2 [6-y]^2\,dy - \pi \int_0^2 [y^2]^2\,dy \\ &= \frac{-\pi(6-y)^3}{3}\bigg|_0^2 - \frac{\pi y^5}{5}\bigg|_0^2 \\ &= \left(\frac{-64\pi}{3} - \frac{-216\pi}{3}\right) - \left(\frac{32\pi}{5} - 0\right) \\ &= \frac{664\pi}{15}. \end{aligned}$$

Whew! That's a lot of things to process. Getting a clear picture of the region is crucial. Even if you never draw the three-dimensional version, understanding the representative rectangles is critical. In this case the representative rectangles did not extend from the bottom to the top curve (i.e., the left to the right), but rather from the y-axis to each curve separately to represent the outer solid and the inner solid.

Caution! We wrote the volume of the solid as the outer minus inner volume:

$$V = \pi \int_0^2 [6-y]^2\,dy - \pi \int_0^2 [y^2]^2\,dy.$$

Since the interval is the same for both integrals, we could have written this as

$$V = \pi \int_0^2 [6-y]^2 - [y^2]^2\,dy$$

where *each individual radius is squared separately*. If you do combine the integrals, you cannot square the difference of the two radii:

$$V \neq \pi \int_0^2 [6-y-y^2]^2\,dy$$

because

$$[6-y]^2 - [y^2]^2 \neq [6-y-y^2]^2.$$

You've been warned!

Look, there are two situations you need to distinguish when multiple curves are used for a solid of revolution. The curves might create two pieces which sum together to create the entire solid in which case you will need to add integrals together to find the volume (see Figure 6.17). Or the curves might be situated so that resulting solid is formed by hollowing out one solid from another in which case you will need to subtract one integral from another (see Figure 6.18). This is why a sketch of the position of the curves relative to the axis of rotation is critical. Let's do a couple more.

EXAMPLE 6.2.7. Consider the region enclosed by the curves $y = \sqrt{x}$, $y = x - 2$, and the x-axis. Rotate this region about the x-axis and find the resulting volume.

Solution. Is this the sum of two integrals or is it difference of two integrals?

First determine where the curves intersect: Obviously $y = \sqrt{x}$ meets the x-axis at $x = 0$ and $y = x - 2$ meets the x-axis at $x = 2$. Now $y = x - 2$ and $y = \sqrt{x}$ intersect when

$$x - 2 = \sqrt{x} \Rightarrow x^2 - 4x + 4 = x \Rightarrow x^2 - 3x + 4 = (x-4)(x+1) = 0$$
$$\Rightarrow x = 4, \ (\text{not } x = -1).$$

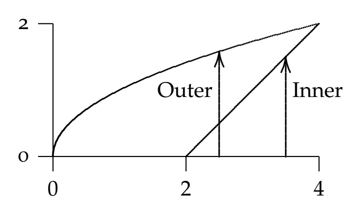

Figure 6.19: Left: The region enclosed by the curves $y = \sqrt{x}$, $y = 2 - x$, and the x-axis. When rotated about the x-axis, one region must be subtracted from the other. Instead of using representative rectangles, we simply indicate the appropriate radii with arrows.

From the sketch in Figure 6.19 if revolved about the x-axis will result in a solid that has been partially hollowed out (a cone has been removed). This requires a difference of integrals. Using Theorem 6.2.1

$$\begin{aligned} V &= \text{Outer Volume} - \text{Inner Volume} \\ &= \pi \int_0^4 [\sqrt{x}]^2\,dx - \pi \int_2^4 [x-2]^2\,dx \\ &= \pi \int_0^4 x\,dx - \pi \int_2^4 [x-2]^2\,dx \\ &= \frac{\pi x^2}{2}\bigg|_0^4 - \frac{\pi(x-2)^3}{3}\bigg|_2^4 \\ &= (8\pi - 0) + \left(\frac{8\pi}{3} - 0\right) \\ &= \frac{16\pi}{3}. \end{aligned}$$

EXAMPLE 6.2.8. Again consider the region enclosed by the curves $y = \sqrt{x}$, $y = x - 2$, and the x-axis. This time rotate the region about the y-axis and find the resulting volume.

Solution. Is this the sum of two integrals or is it difference of two integrals? Since the rotation is about the y-axis, the radii of the respective regions are horizontal, see Figure 6.20. This is again a difference of two integrals.

Translating the curves into functions of y we have $x = y^2$, $x = y + 2$, and $y = 0$ (the x-axis). The curves intersect the x-axis at $y = 0$. We've seen that the line and square root function meet when $x = 4$ since $x = y + 2$ there, then the y-coordinate of the intersection is $y = 2$.

Using Theorem 6.2.1

$$\begin{aligned} V = \text{Outer Volume} - \text{Inner Volume} &= \pi \int_0^2 [y+2]^2\,dy - \pi \int_0^2 [y^2]^2\,dy \\ &= \frac{\pi(y+2)^3}{3}\bigg|_0^2 - \frac{\pi y^5}{5}\bigg|_0^2 \\ &= \left(\frac{64\pi}{3} - \frac{8\pi}{3}\right) - \left(\frac{32\pi}{5} - 0\right) \\ &= \frac{184\pi}{15}. \end{aligned}$$

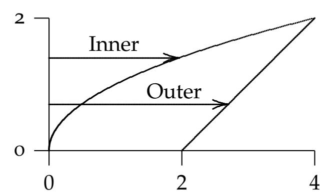

Figure 6.20: The region enclosed by the curves $y = \sqrt{x}$, $y = 2 - x$, and the x-axis and representative radii. When rotated about the y-axis, one region must be subtracted from the other.

EXAMPLE 6.2.9. Consider the region enclosed by the curves $y = e^x$, $x = 1$ and the y-axis. Rotate the region about the y-axis and find the resulting volume.

Solution. Is this the sum of two integrals or is it difference of two integrals? Since the rotation is about the y-axis, the radii of the respective regions are horizontal, see Figure 6.21. This is again a difference of two integrals.

Translating the curves into functions of y we see $y = e^x$ becomes $x = \ln y$. Notice that $x = \ln y$ meets the y-axis at 1 and it meets the line $x = 1$ at $y = e$.

The solid formed by the region between the y-axis and the curve $x = \ln y$ must be subtracted from the ordinary cylinder formed by rotating the line $x = 1$ about the y-axis. This cylinder has radius 1 and height e so its volume is $v = \pi(1)^2 e = \pi e$. (We could also find this by integrating, but why bother.) So Using Theorem 6.2.1

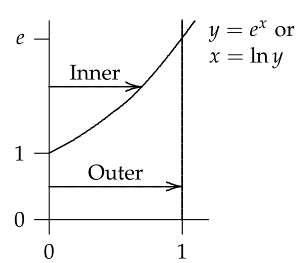

Figure 6.21: The region enclosed by the curves $y = e^x$, $x = 1$ and the y-axis. Rotate the region about the y-axis and find the resulting volume.

$$V = \text{Outer Volume} - \text{Inner Volume} = \pi e - \pi \int_1^e [\ln y]^2\,dy = ???$$

Since we don't know an antiderivative for $(\ln x)^2$ we are stuck. (By the way, notice that the limits of integration for the integral are from 1 to e, not 0 to e.) So we will have to invent another method of finding such volumes if we are to solve this problem. This shows the importance of having a wide-variety of methods for solving a single problem.

YOU TRY IT 6.10. Let R be the *entire* region enclosed by $y = x^2$ and $y = 2 - x^2$ in the upper half-plane. Sketch the region. Rotate R about the x-axis and find the resulting volume. (Answer: $\frac{16}{3}\pi$.)

YOU TRY IT 6.11. Let R be the region enclosed by $y = 2x$ and $y = x^2$.

(*a*) **Rotation about the x-axis.** Find the volume of the hollowed-out solid generated by revolving R about the x-axis. (Answer: $64\pi/15$)

(*b*) **Rotation about the y-axis.** Find the volume of the solid generated by revolving R about the y-axis by using the disk method and integrating along the y-axis. (Answer: $8\pi/3$)

WEBWORK: Click to try Problems 83 through 86. Use GUEST login, if not in my course.

6.3 *Optional: Rotation About Other Axes*

It is relatively easy to adapt the disk method to finding volumes of solids of revolution using other horizontal or vertical axes. The key steps are to determine the radii of the slices and express them in terms of the correct variable. A couple of examples should give you the idea.

EXAMPLE 6.3.1. Consider the region enclosed by the curves $y = x^2 - 2x$ and $y = 3$. Rotate this region about the line $y = 3$ and find the resulting volume.

Solution. The two curves meet when

$$x^2 - 2x = 3 \Rightarrow x^2 - 2x - 3 = (x-3)(x+1) = 0 \Rightarrow x = 3,\ -1.$$

The parabola and the line are easy to sketch; see Figure 6.22 on the left.

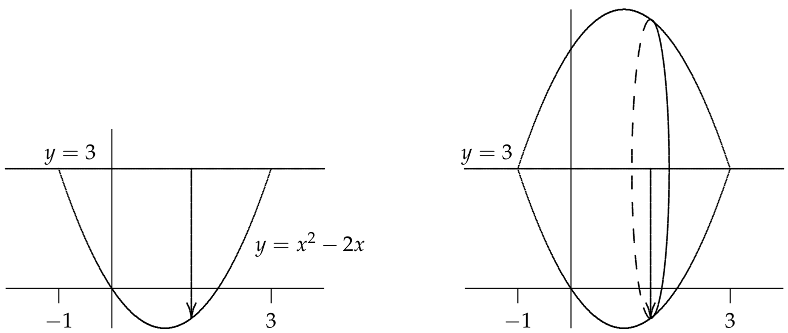

Figure 6.22: Left: The region enclosed by the curves $y = x^2 - 2x$ and $y = 3$. Since the axis of revolution is $y = 3$, a representative radius extends from the line $y = 3$ to the curve $y = x^2 - 2x$. Right: The resulting solid of revolution about the line $y = 3$.

A representative radius extends from the line $y = 3$ to the curve $y = x^2 - 2x$. The length of a radius is the difference between these two values, that is, the radius of a circular cross-section perpendicular to the line $y = 3$ is

$$r = 3 - (x^2 - 2x) = 3 + 2x - x^2.$$

Since a cross-section is a circle, its area is

$$A = A(x) = \pi r^2 = \pi(3 + 2x - x^2)^2.$$

Since we know the cross-sectional area, we can use Theorem 6.1.1 to find the volume

$$\begin{aligned} V = \int_a^b A(x)\,dx &= \pi \int_{-1}^{3} (3 + 2x - x^2)^2\,dx \\ &= \pi \int_{-1}^{3} 9 + 12x - 2x^2 - 4x^3 + x^4)^2\,dx \\ &= \pi \left(9x + 6x^2 - \frac{2x^3}{3} - x^4 + \frac{x^5}{5} \right) \Bigg|_{-1}^{3} \\ &= \pi \left[\left(27 + 54 - 18 - 81 + \frac{243}{5} \right) - \left(-9 + 6 + \frac{2}{3} - 1 - \frac{1}{5} \right) \right] \\ &= \frac{512\pi}{15}. \end{aligned}$$

EXAMPLE 6.3.2. Consider the region enclosed by the curves $x = y^2$ and $x = 2 - y^2$. Rotate this region about the line $x = 3$ and find the resulting volume.

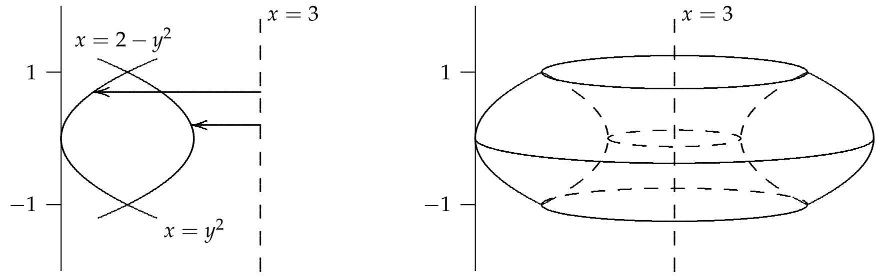

Figure 6.23: Left: The region enclosed by the curves $x = y^2$ and $x = 2 - y^2$ and two representative radii emanating from the axis of rotation $x = 3$. Right: The resulting 'hollowed out' solid of revolution about the line $x = 3$ looks like a 'doughnut.'

Solution. The two curves meet when

$$y^2 = 2 - y^2 \Rightarrow 2y^2 = 2 \Rightarrow y = \pm 1.$$

The region is easy to sketch; see Figure 6.23 on the left.

An 'outer' representative radius extends from the line $x = 3$ to the curve $x = y^2$ and an 'inner' representative radius extends from the line $x = 3$ to $x = 2 - y^2$. The length of a representative radius is the difference between the corresponding pairs of values. The outer radius is $R = 3 - y^2$ and the inner radius is $3 - (2 - y^2) = 1 + y^2$. The integration will take place along the y-axis on the interval $[-1, 1]$ because the disks are horizontal when points are rotated about the line $x = 3$. Since we know the cross-sections are circles, we can use Theorem 6.1.1 to find the volume

$$\begin{aligned} V = \text{Outer Volume} - \text{Inner Volume} &= \pi \int_{-1}^{1} (3 - y^2)^2 \, dy - \pi \int_{-1}^{1} (1 + y^2)^2 \, dy \\ &= \pi \int_{-1}^{1} 9 - 6y^2 + y^4 \, dy - \pi \int_{-1}^{1} 1 + 2y^2 + y^4 \, dy \\ &= \pi \int_{-1}^{1} 8 - 8y^2 \, dy \\ &= \pi \left(8y - \frac{8y^3}{3} \right) \Big|_{-1}^{1} \\ &= \pi \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] \\ &= \frac{32\pi}{3}. \end{aligned}$$

Because the interval of integration was the same for both the inner and outer volumes, we are able to combine the two integrals which greatly simplified the integration and evaluation. If you understand this problem, you are in good shape.

YOU TRY IT 6.12. Let R be the region enclosed by $y = x^3$, $x = 2$, and the x-axis. Draw the region.

(*a*) Rotate R about the x-axis and find the resulting volume. (Answer: $128\pi/7$)

(*b*) Rotate R about the line $y = 9$ and find the resulting volume. (Answer: $376\pi/7$)

(*c*) Rotate R about the line $x = 2$ and find the resulting volume. (Answer: $16\pi/5$)

YOU TRY IT 6.13. Draw the region R in the first quadrant enclosed by $y = x^2$, $y = 2 - x$ and the x-axis.

(*a*) Rotate R about the x-axis and find the resulting volume. [Is it the sum of two integrals or outside minus inside?]

(*b*) Rotate R about the y-axis and find the resulting volume. [Is it the sum of two integrals or outside minus inside?]

YOU TRY IT 6.14. Let S be the region in the first quadrant enclosed by $y = x^2$, $y = 2 - x$ and the y-axis.

(*a*) Rotate S about the x-axis. [Is it the sum of two integrals or outside minus inside?]

(*b*) Rotate S about the y-axis. [Is it the sum of two integrals or outside minus inside?]

YOU TRY IT 6.15. Let R be the region in the upper half-plane bounded by $y = \sqrt{x+2}$, the x-axis and the line $y = x$. Find the volume resulting when R is rotated around the x-axis. Remember: Outside minus inside. How many integrals do you need?

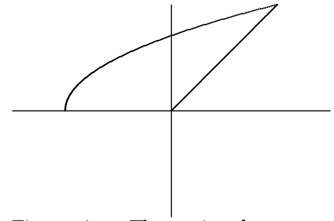

Figure 6.24: The region for YOU TRY IT 6.15 .

YOU TRY IT 6.16. A small canal bouy is formed by taking the region in the first quadrant bounded by the y-axis, the parabola $y = 2x^2$, and the line $y = 5 - 3x$ and rotating it about the y-axis. (Units are feet.) Find the volume of this bouy. (Answer: 2π cubic feet.)

YOU TRY IT 6.17. **Just set up the integrals** for each of the following volume problems. Simplify the integrands where possible. Use figures below.

(*a*) R is the region enclosed by $y = x^2$, $y = x + 2$, and the y-axis in the first quadrant. Rotate R about the x-axis.

(*b*) Same R, but rotate around the y-axis.

(*c*) Same R, but rotate around the line $y = 4$.

(*d*) S is the region enclosed by $y = -x^2 + 2x + 3$, $y = 3x - 3$, the y-axis, and the x-axis in the first quadrant. Rotate S about the x-axis.

(*e*) T is the region enclosed by $y = -x^2 + 2x + 3$, $y = 3x - 3$, and the x-axis in the first quadrant. Rotate T about the x-axis.

(*f*) R is the region enclosed by $y = \frac{1}{2}x^2 + 2$ and $y = x^2$. Rotate R about the x-axis.

(*g*) Same R, but rotate around the line $y = 4$.

(*h*) S is the region enclosed by $y = \frac{1}{2}x^2 + 2$ and $y = x^2$ in the first quadrant. Rotate S about the y-axis.

(*i*) T is the region enclosed by $y = \sqrt{x-2}$, $y = 4 - x$, the y-axis and the x-axis. Rotate T about the y-axis.

(*j*) V is the region enclosed by $y = \sqrt{x-2}$, $y = 4 - x$, and the x-axis. Rotate V about the y-axis.

(*k*) Same V, but rotate around the x-axis.

(*l*) Hard: Same V, but rotate around the line $y = 2$.

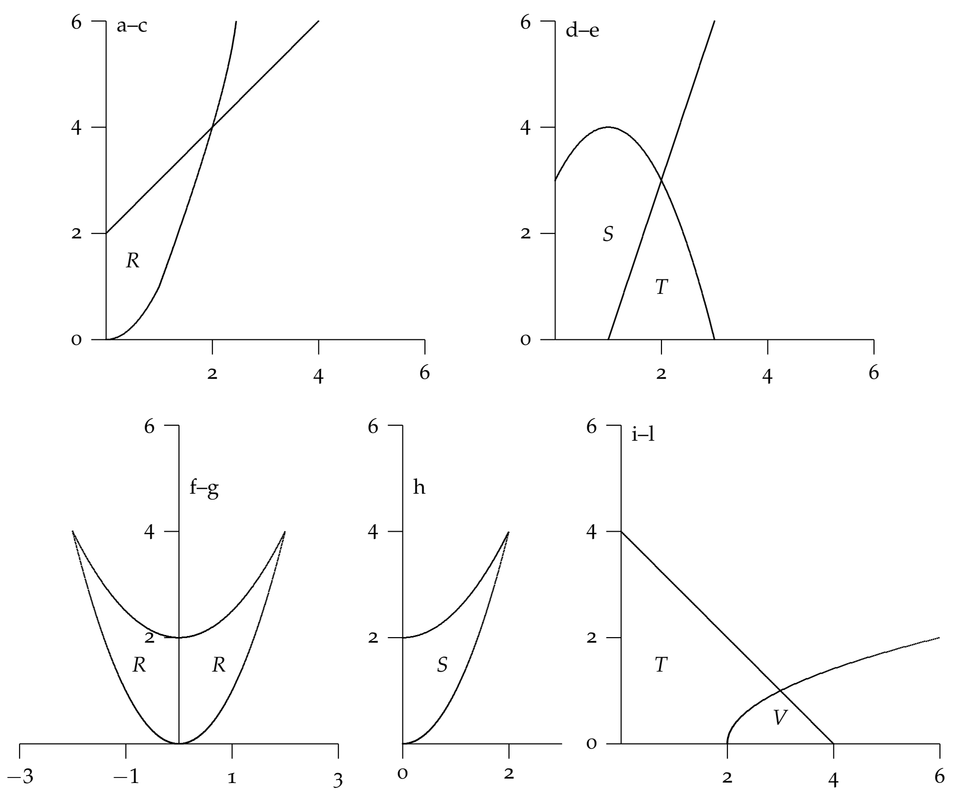

Figure 6.25: The regions for YOU TRY IT 6.17 .

Solution. Here are the solutions **before the integands have been simplified**. Remember, you should simplify each integrand before actually integrating.

(*a*) $\pi \int_0^2 (x+2)^2\,dx - \pi \int_0^2 (x^2)^2\,dx$

(*b*) $\pi \int_0^4 (\sqrt{y})^2\,dy - \pi \int_2^4 (y-2)^2\,dy$

(*c*) $\pi \int_0^2 (4-x^2)^2\,dx - \pi \int_0^2 (4-(x+2))^2\,dx$

(*d*) $\pi \int_0^2 (-x^2+2x+3)^2\,dx - \pi \int_1^2 (3x-3)^2\,dx$

(*e*) $\pi \int_1^2 (3x-3)^2\,dx + \pi \int_2^3 (-x^2+2x+3)^2\,dx$

(*f*) Homework

(*g*) Homework

(*h*) $\pi \int_0^4 (\sqrt{y})^2\,dy - \pi \int_2^4 \left(\sqrt{2y-4}\right)^2\,dy$

(*i*) $\pi \int_0^1 (y^2+2)^2\,dy + \pi \int_1^4 (4-y)^2\,dy$

(*j*) $\pi \int_0^1 (4-y)^2\,dy - \pi \int_0^1 (y^2+2)^2\,dy$

(*k*) $\pi \int_2^3 (\sqrt{x-2})^2\,dx + \pi \int_3^4 (4-x)^2\,dx$

(*l*) $\pi \int_2^3 2^2\,dx - \pi \int_2^3 (2-\sqrt{x-2})^2\,dx + \left[\pi \int_3^4 2^2\,dx - \pi \int_3^4 2^2(2-(4-x))^2\,dx\right]$

YOU TRY IT 6.18 (Rotation about the x-axis). Let R be the region in the first quadrant enclosed by $y = \sqrt{x-1}$, $y = x-7$ and the x-axis. Sketch the region. Rotate R about the x-axis and find the resulting volume. (Answer: $63/2\pi$.)

YOU TRY IT 6.19. Let R be the *entire* region enclosed by $y = x^2$ and $y = 2 - x^2$ in the upper half-plane. Sketch the region. Rotate R about the x-axis and find the resulting volume. (Answer: $\frac{16}{3}\pi$.)

YOU TRY IT 6.20 (Rotation about the y-axis). Let R be the region enclosed by the x-axis, $y = \sqrt{x}$, and $y = 2 - x$. Rotate R about the y-axis and find the volume. (Answer: $32\pi/15$.)

YOU TRY IT 6.21 (Rotation about a line parallel to the x-axis). Find the volume of the hollowed-out solid generated by revolving R about the line $y = 4$. (Answer: $32\pi/5$)

YOU TRY IT 6.22 (Rotation about a line parallel to the x-axis). Let R be the region enclosed by $y = 1 - x^2$ and the x-axis. Rotate R about the line $y = 2$ and find the resulting volume. (Answer: $64\pi/15$.)

YOU TRY IT 6.23. **Extra Fun.**

(*a*) Let R be the region enclosed by $y = \arctan x$, $y = \pi/4$, and the y-axis. Find the area of R. Hint: Only integration along one axis is possible at this point. (Answer: $\ln\sqrt{2} = 0.5\ln 2$)

(*b*) Rotate R about the y-axis and find the resulting volume. Use a trig identity for $\tan^2\theta$. (Answer: $\pi - \pi^2/4$)

YOU TRY IT 6.24. Here are a few more:

(*a*) Let $y = \frac{1}{x}$ on $[1, a]$, where $a > 1$. Let R be the region under the curve over this interval. Rotate R about the x-axis. What value of a gives a volume of $\pi/2$? (Answer: $a = 2$)

(*b*) What happens to the volume if $a \to \infty$? Does the volume get infinitely large? Use limits to answer the question.

(*c*) Instead rotate R about the y-axis. What value of a gives a volume of $\pi/2$? This is particularly easy to do by shells! (Answer: $a = 5/4$)

(*d*) Instead rotate R about the line $y = -1$. If we let $a = e$ what is the resulting volume? (Answer: $3\pi - \frac{\pi}{e}$)

YOU TRY IT 6.25. Let R be the region enclosed by $y = \sqrt{x}$, $y = 2$, and the y-axis.

(*a*) Rotate R about the x-axis and find the resulting volume. (Answer: 8π)

(*b*) Rotate R about the line $y = 2$ and find the resulting volume. (Answer: $8\pi/3$)

(*c*) Rotate R about the line $y = 4$ and find the resulting volume. (Answer: $40\pi/3$)