Application: Arc Length

7.1 *The General Problem*

The Riemann integral has a wide variety of applications. In this section, using the 'subdivide and conquer' strategy we will show how it can be used to determine the lengths of certain curves.

Figure 7.1: A hanging chain forms a catenary. `http://en.wikipedia.org/wiki/Catenary`

EXAMPLE 7.1.1. Verizon is hanging fiber optic cable around Geneva. It wants to know how much cable it will need. Since the cable must be hung with some slack (why?), the engineers can't simply measure the distance between the poles to know how much cable to use.

The curve that an idealized hanging chain or cable assumes when supported at its ends and acted on only by its own weight is called a *catenary*. The equation for the graph of a catenary curve is a hyperbolic cosine function

$$a \cosh\left(\frac{x}{a}\right) = \frac{a}{2}(e^{x/a} + e^{-x/a}),$$

where a is a constant. How can Verizon use this equation to find the length of the cable it needs?

Let's try to attack this problem in a more general fashion. Suppose $y = f(x)$ is a continuous function on the closed interval $[a, b]$. Find the length of the graph of $y = f(x)$ on this interval. This length is called the *arc length* of the curve.

Think about how we approached the area problem. We used the only figures for which we had an area formula (rectangles) and used those figures to approximate the area under a curve. Well, the only curve whose length we are certain of is a straight line segment.[1] If the segment has endpoints (x_1, y_1) and (x_2, y_2), then the length of the segment is given by the distance formula

[1] You might protest that you know the circumference of a circle: $2\pi r$. But *why* is that formula true?

$$\text{Segment Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

So somehow we must use line segments (the only curves we know how to measure) to obtain the length of a more general continuous curve.

Well, let's do the usual thing: 'subdivide and conquer.' Suppose f is continuous on $[a, b]$. Given a ruler, we might mark off successive points $Q_1,\ Q_2,\ \ldots,\ Q_n$ along the curve and take the resulting polygonal arc as an approximation to the curve (see Figure 7.2). We could measure the length of each segment of the polygonal arc with our ruler (or the distance formula) and then summing these values together we would have an approximation of the length of the curve. This idea (and the lack of a precise definition of length for curves) motivates the following process.

Assume that f is continuous over the closed, bounded interval $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[a, b]$ into n equal width subintervals. For $i = 1$ to n let $Q_i = (x_i, f(x_i))$ be the corresponding set of points on the graph of f. Then the **polygonal arc** from Q_0 to Q_n is just the sequence of line segments

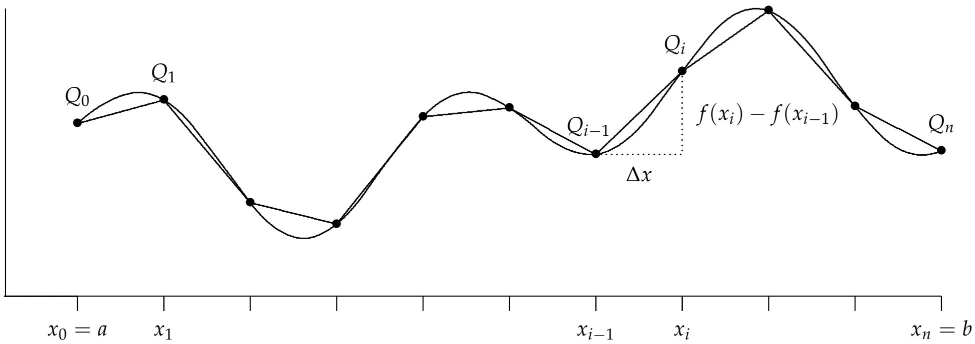

Figure 7.2: A curve $y = f(x)$ and a polygonal approximation using a regular partition.

Q_0Q_1, Q_1Q_2, $\dots$, $Q_{n-1}Q_n$. The length of this polygonal arc is just the sum of the lengths of the individual segments. Using the Pythagorean theorem the length of the ith segment is

$$\text{Length(Segment } i) = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}. \tag{7.1}$$

We can simplify (7.1) in two ways. As usual, we let $\Delta x = x_i - x_{i-1}$. Next, by the Mean Value Theorem (yet once again!), if we assume that f is also *differentiable* on $[a, b]$, then

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i)$$

for some point c_i between x_{i-1} and x_i. Consequently,

$$f(x_i) - f(x_{i-1}) = f'(c_i) \cdot [x_i - x_{i-1}] = f'(c_i)\Delta x.$$

Making both of these changes to (7.1) we get

$$\text{Length(Segment } i) = \sqrt{(\Delta x)^2 + [f'(c_i)\Delta x]^2}. = \sqrt{1 + [f'(c_i)]^2}\Delta x. \tag{7.2}$$

Adding the lengths of all n line segments together we have

$$\text{Length of Curve} \approx \sum_{i=1}^{n} \text{Length(Segment } i) = \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} \cdot \Delta x.$$

Notice that we now have a Riemann sum! To improve the approximation we do the standard thing: We let the number of polygonal pieces get large and take the limit. We find

$$\text{Length of Curve} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} \cdot \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

To be certain that this limit of the Riemann sums is, in fact, the definite integral we need to know that

$$\sqrt{1 + (f'(x))^2}$$

is continuous. It will be, if we assume that $f'(x)$ is continuous. With these conditions, we find

THEOREM 7.1.2 (Arc Length Formula). If f is differentiable and f' is continuous on the closed interval $[a, b]$, then the **arc length** of f on $[a, b]$ is

$$\text{Arc Length of } f = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

EXAMPLE 7.1.3. Find the arc length of f over $[2, 6]$ for $f(x) = \frac{4}{3}x^{3/2}$.

Solution. First determine the derivative: $f'(x) = 2x^{1/2}$ on $[2,6]$. So by Theorem 7.1.2

$$\begin{aligned}\text{Arc Length of } f = \int_a^b \sqrt{1+[f'(x)]^2}\,dx &= \int_2^6 \sqrt{1+(2x^{1/2})^2}\,dx \\ &= \int_2^6 \sqrt{1+4x}\,dx \\ &= \frac{2(1+4x)^{3/2}}{12}\bigg|_2^6 \\ &= \frac{(125-27)}{6} = \frac{49}{3}.\end{aligned}$$

Notice that we used a 'mental adjustment' in the integration: $u = 1+4x$.

EXAMPLE 7.1.4. Find the arc length of f over $[0, \pi/4]$ for $f(x) = \ln(\cos x)$.

Solution. Determine the derivative $f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$ on $[0, \pi/4]$. So by Theorem 7.1.2

$$\begin{aligned}\text{Arc Length of } f = \int_a^b \sqrt{1+[f'(x)]^2}\,dx &= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx \\ &= \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx \\ &= \int_0^{\pi/4} \sec x\,dx \\ &= \ln|\sec x + \tan x|\Big|_0^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln 1 \\ &= \ln(\sqrt{2}+1).\end{aligned}$$

EXAMPLE 7.1.5 (Circumference of a Circle). Find the arc length (circumference) of a circle of radius r.

Solution. To make things easy we will assume that the circle is centered at the origin so that its equation is $f(x) = \sqrt{r^2 - x^2}$ on the interval $[-r, r]$. Actually this is the equation for the *upper* semi-circle of radius r. We'd need to use the negative root to get the lower half of the circle. Ok, we can deal with that by simply doubling the answer.

The derivative

$$f'(x) = \frac{\frac{2}{2}(-2x)}{\sqrt{r^2-x^2}} = -\frac{x}{\sqrt{r^2-x^2}}$$

and is not defined at r and $-r$ where the denominator is 0. To avoid this, let's use one-twelfth of a circle (30°) by restricting the domain of f to $\left[0, \frac{r}{2}\right]$. See Figure 7.3.

Since we are using only one-twelfth of the circle, we need to multiply the an-

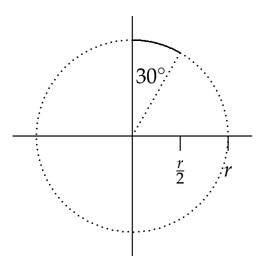

Figure 7.3: One-twelfth of a circle (30°) of radius r.

swer by 12, so

$$\begin{aligned}
\text{Circumfrence of Circle} &= 12\int_0^{r/2} \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\
&= 12\int_0^{r/2} \sqrt{\frac{(r^2-x^2)+x^2}{r^2-x^2}}\,dx \\
&= 12\int_0^{r/2} \sqrt{\frac{r^2}{r^2-x^2}}\,dx \\
&= 12\int_0^{r/2} \frac{r}{\sqrt{r^2-x^2}}\,dx \\
&= 12r\arcsin\left(\frac{x}{r}\right)\Bigg|_0^{r/2} \\
&= 12r\left[\arcsin\left(\frac{1}{2}\right)-\arcsin(0)\right] \\
&= 12r\left(\frac{\pi}{6}\right) \\
&= 2\pi r.
\end{aligned}$$

Awesome! We have now proven that the circumference of a circle of radius r is $2\pi r$. Earlier we showed that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Interestingly, we have not yet shown that the area of a circle of radius r is πr^2. The reason for this is that to determine the area we need to calculate

$$\int_{-r}^{r} \sqrt{r^2-x^2}\,dx$$

but we do not yet know an antiderivative of $\sqrt{r^2-x^2}$. (Look it up in a table! Find the area of a circle using this.)

EXAMPLE 7.1.6 (Length of a Line Segment). Our arc length formula better work for the length of an ordinary line segment. Let's check. Find the length of the segment between $(3,4)$ and $(8,-6)$.

Solution. Since the segment is part of a non-vertical line, its equation has the form $y = f(x) = mx+b$, so $f'(x) = m$. The slope of the segment is

$$m = \frac{4-(-6)}{3-8} = \frac{10}{-5} = -2.$$

Using Theorem 7.1.2 we find

$$\begin{aligned}
\text{Arc Length of Segment} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx &= 12\int_3^8 \sqrt{1+(2)^2}\,dx \\
&= 12\int_3^8 \sqrt{5}\,dx \\
&= \sqrt{5}x\Big|_3^8 \\
&= \sqrt{5}(8-3) \\
&= 5\sqrt{5}.
\end{aligned}$$

Using the distance formula, the length of the segment is

$$\sqrt{(8-3)^2+(-6-4)^2} = \sqrt{25+100} = \sqrt{125} = 5\sqrt{5},$$

which is the same answer we got using the arc length formula. Phew!

EXAMPLE 7.1.7 (Simplifying). Let $f(x) = \frac{1}{10}x^5 + \frac{1}{6}x^{-3}$ on $[1, 2]$. Find the arc length.

Solution. Ok, time to 'fess up. There are not many arc length integrals we can do. This is one of them, but it requires some algebraic manipulation. Once you see this you should recognize similar integrals.

The derivative is

$$f'(x) = \frac{1}{2}x^4 - \frac{1}{2}x^{-4}.$$

Notice that the exponents are negatives of each other. This is important because when we square $f'(x)$,

$$[f'(x)]^2 = \left(\frac{1}{2}x^4 - \frac{1}{2}x^{-4}\right)^2 = \frac{1}{4}x^8 - \frac{1}{2} + \frac{1}{4}x^{-8},$$

We get a constant of $-\frac{1}{2}$ for the middle term. Consequently, when we add one to this, we get $\frac{1}{2}$ for the middle term,

$$1 + [f'(x)]^2 = 1 + \frac{1}{4}x^8 - \frac{1}{2} + \frac{1}{4}x^{-8} = \frac{1}{4}x^8 + \frac{1}{2} + \frac{1}{4}x^{-8} = \left(\frac{1}{2}x^4 + \frac{1}{2}x^{-4}\right)^2,$$

which is again a perfect square. Now we are ready to calculate the arc length using Theorem 7.1.2 we find

$$\begin{aligned} \text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx &= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2}x^{-4}\right)^2}\, dx \\ &= \int_1^2 \frac{1}{2}x^4 + \frac{1}{2}x^{-4}\, dx \\ &= \frac{1}{10}x^5 - \frac{1}{6}x^{-3}\Big|_1^2 \\ &= \left(\frac{32}{10} - \frac{1}{48}\right) - \left(\frac{1}{10} - \frac{1}{6}\right) \\ &= \frac{779}{240}. \end{aligned}$$

Messy, but doable! (Notice the similarity between the original function f and the antiderivative of $\sqrt{1 + [f'(x)]^2}$ which is characteristic of this type of problem.)

It's a sad fact, arc lengths involve complicated integrands so without additional integration methods (coming soon!) there are not many arc lengths that we can compute. Here are a few more we'd like to do.

EXAMPLE 7.1.8. What about the arc length of a parabola, say $f(x) = x^2$ on $[0, 1]$. Check that

$$\text{Arc Length} = \int_0^1 \sqrt{1 + 4x^2}\, dx,$$

but we do not yet have an antiderivative for this problem. Similarly If $g(x) = \sin x$ on $[0, \pi]$, then

$$\text{Arc Length} = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

Or if $f(x) = e^x$ on $[0, 1]$, then

$$\text{Arc Length} = \int_0^1 \sqrt{1 + e^{2x}}\, dx.$$

If you know how to use *Wolfram Alpha*, you might try these problems. But for the moment, we are stuck!

YOU TRY IT 7.1. Find the arc lengths of the following functions over the given intervals.

(*a*) $f(x) = \frac{1}{3}x^{3/2}$ on $[0, 8]$. (Answer: $\frac{8}{3}(2^{3/2} - 1)$.)

(*b*) $f(x) = \frac{x^4}{4} + \frac{1}{8x^2}$ on $[1, 2]$. (Answer: $\frac{123}{32}$.)

(*c*) $g(x) = x^3 + \frac{1}{12x}$ on $[1, 3]$. (Answer: $\frac{469}{18}$.)

(*d*) $h(x) = \cosh x$ on $[0, \ln 2]$. Remember, $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$. (Answer: $\frac{3}{4}$.)

YOU TRY IT 7.2. Here are few additional problems

(*a*) Find the *exact* arc length of $f(x) = 5 - 2x^{3/2}$ on the interval $[0, 11]$. (Answer: 74).

(*b*) Find the arc length of $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ on the interval $[1, 2]$. Simplify the integrand! (Answer: $\frac{33}{16}$)

(*c*) Find the arc length of $f(x) = \ln \sec x$ on $[0, \pi/4]$. Use a trig identity! (Answer: $\ln(\sqrt{2} + 1)$)

YOU TRY IT 7.3 (Extra Credit). Solve the problem in Example 7.1.1 on the interval $[-a, a]$.

WEBWORK: Click to try Problems 91 through 94.