21. Determine whether the series $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ converges.

Solution. (a) Use the Root Test. (b) The given series has powers and exponents. (c) The terms $a_k = \frac{k^2}{2^k}$ are positive. (d) Notice that

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(n^2)^{1/n}}{2^{n/n}} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{2} \stackrel{\text{Key Lim}}{=} \frac{1^2}{2} = \frac{1}{2} < 1.$$

(e) Since $r < 1$, the series converges by the Root Test.

28. Determine whether the series $\sum_{k=1}^{\infty} \frac{k^2+k-1}{k^4+4k^2-3}$ converges.

Solution. (a) Use Limit Comparison. (b) The given series looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (c) The terms $a_n = \frac{n^2+n-1}{n^4+4n^2-3}$ and $b_n = \frac{1}{n^2}$ are always positive. (d) Notice that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n-1}{n^4+4n^2-3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^4+n^3-n}{n^4+4n^2-3} = \lim_{n\to\infty} \frac{1+\frac{1}{n}-\frac{1}{n^3}}{1-\frac{4}{n^2}-\frac{3}{n}} = 1.$$

(e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), and $0 < L < \infty$, so so $\sum_{k=1}^{\infty} \frac{k^2+k-1}{k^4+4k^2-3}$ converges by the Limit Comparison Test.

34. Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{3^k-2^k}$ converges.

Solution. (a) Use Limit Comparison. (b) The given series looks a lot like the geometric series $\sum_{k=1}^{\infty} \frac{1}{3^k}$. (c) The terms $a_k = \frac{1}{3^k-2^k}$ and $b_n = \frac{1}{3^k}$ are always positive. (d) Notice that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{3^n-2^n} \cdot \frac{3^n}{1} = \lim_{n\to\infty} \frac{3^n}{3^n-2^n} = \lim_{n\to\infty} \frac{1}{1-(\frac{2}{3}^n} = \frac{1}{1-0} = 1.$$

(e) The geometric series $sum_{k=1}^{\infty} \frac{1}{3^k}$ converges ($|r| = \frac{1}{3} < 1$), and $0 < L < \infty$, so so $\sum_{k=1}^{\infty} \frac{1}{3^k-2^k}$ converges by the Limit Comparison Test.

38. Determine whether the series $\sum_{k=2}^{\infty} \frac{1}{(k\ln k)^2}$ converges.

Solution. (a) Use Limit Comparison. (b) The given series looks a lot like the geometric series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. (c) The terms $a_k = \frac{1}{(k\ln k)^2}$ and $b_k = \frac{1}{k^2}$ are always positive. (d) Notice that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(n\ln n)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2\ln^2 n} = \lim_{n\to\infty} \frac{1}{\ln^2 n} = 0.$$

(e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), and $L = 0$, so $\sum_{k=1}^{\infty} \frac{1}{(k\ln k)^2}$ converges by the Limit Comparison Test.

65. Determine whether the series $\sum_{k=1}^{\infty} \tan\frac{1}{k}$ converges.

Solution. (a) Use Limit Comparison. (b) The given series looks a lot like the geometric series $\sum_{k=1}^{\infty} \frac{1}{k}$. (c) The terms $a_k = \tan\frac{1}{k}$ and $b_k = \frac{1}{k}$ are always positive. (d) Notice that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\tan\frac{1}{x}}{\frac{1}{x}} \stackrel{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x}) \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \sec^2(\tfrac{1}{x}) = \sec^2(0) = 1.$$

(e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p = 1 \leq 1$), and $0 < L < \infty$, so $\sum\limits_{k=1}^{\infty} \tan \frac{1}{k}$ diverges by the Limit Comparison Test.

12. Determine whether the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\sqrt{k}}$ converges.

Solution. (a-b) Use Alternating Series test with $a_k = \frac{1}{\sqrt{k}}$ because the series is alternating. (c) The terms $a_k = \frac{1}{\sqrt{k}}$ are always positive. (d) Check the two conditions of the test.

1. Decreasing? Use the derivative. Let $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$. Then $f'(x) = -\frac{x^{-3/2}}{2} < 0$ for x in $[1, \infty)$. So the function and corresponding sequence are decreasing.

2. $\lim\limits_{k \to \infty} a_k = \lim\limits_{k \to \infty} \dfrac{1}{\sqrt{k}} = 0.$

(e) Since the series satisfies the two conditions, by the Alternating Series test, $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\sqrt{k}}$ converges.