My Office Hours: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. **Math Intern:** Sun: 2:00–5:00, 7:00–10pm; Mon thru Thu: 3:00–5:30 and 7:00–10:30pm in Lansing 310. Website: `http://math.hws.edu/~mitchell/Math131F15/index.html`.

☕ *Practice.* **Review** Section 8.6 on Alternating Series 649–652 and Absolute/Conditional Convergence. Skip the subsection on remainders. But do read pages 654–655 on Absolute Convergence. See the nice summary chart on page 656.

1. (*a*) Comparison and Limit Comparison Tests. Page 648 #27, 29, 33, 35 and 44.
(*b*) Alternating Series Test. Page 657 #11, 13, 15, 19, and 21.
(*c*) If we get this far: Absolute and Conditional Convergence. Page 657 #45, 47, 49.

For #44 use the comparison test. What do you know about the size of $\sin^2 k$?

The More Recent Tests

1. Limit Comparison Test. Assume that $a_n > 0$ and $b_n > 0$ for all n (or at least all $n \geq k$) and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

(1) If $0 < L < \infty$ (i.e., L is a positive, *finite* number), then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ *both* converge or *both* diverge.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(3) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Direct Comparison Test. Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(*a*) If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
(*b*) If $0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

If the bigger series converges, so does the smaller series.
If the smaller series diverges, so does the bigger series.

3. The Alternating Series Test. Assume $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the following two conditions hold:

(*a*) The terms a_n are (eventually) decreasing (non-increasing), that is, $a_{n+1} \leq a_n$ for all n (or for all $n > N$).
(*b*) $\lim_{n\to\infty} a_n = 0$

4. Absolute Convergence Test. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

Hand In At Lab—Be Neat and Careful. Use the Model Methods.

MODEL 1: Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{n - \frac{1}{2}}$ converges or diverges.

Solution. This problem is easiest with the Limit Comparison Test. (See the handout from last time.) But here's how to (a) Use the Direct Comparison Test. (b) The given series looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-series test). So we think that $\sum_{n=1}^{\infty} \frac{2}{n-\frac{1}{2}}$ diverges. (c–d) So in the Direct Comparison Test we these terms to be LARGER than the comparison series. Notice that $n - \frac{1}{2} < n$ so taking reciprocals changes the direction of the inequality and we have

$$\frac{1}{n - \frac{1}{2}} > \frac{1}{n} \text{ so that } \frac{2}{n - \frac{1}{2}} > \frac{2}{n} \geq 0.$$

(e) Since $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges (p-series test, $p = 1 \leq 1$), it follows that $\sum_{n=1}^{\infty} \frac{2}{n-\frac{1}{2}}$ diverges by the Direct Comparison Test.

Model 2: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+6n}$ converges.

Solution. (a) Use Direct Comparison. (b) The given series looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (c) The terms $a_n = \frac{\cos^2(n)}{n^2+6n}$ and $b_n = \frac{1}{n^2}$ are always positive. (d) Notice that $0 < \frac{\cos^2(n)}{n^2+6n} < \frac{1}{n^2}$ because $\cos^2 < 1$ and the denominator $n^2 + 6n > n^2$. (e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+6n}$ is SMALLER than the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, so it also converges by the Comparison Test.

☞Limit Comparison is not possible because $\cos^2(n)$ does not have a limit.

Model 3: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+6n}$ converges.

Solution. (a–b) Use the Alternating Series test since the series is alternating. (c) Here $a_n = \frac{n}{n^2+6n}$. Check the two conditions.

(1) Decreasing? Let $f(x) = \frac{x}{x^2+6x}$. Then $f'(x) = \frac{x^2+6x-x(2x+6)}{(x^2+6x)^2} = \frac{x^2+6x-2x^2-6x}{(x^2+6x)^2} = \frac{-x^2}{(x^2+6x)^2} < 0$, so the function and the sequence are decreasing.

(2) And $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+6n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{6}{n}} = 0$.

(e) By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+6n}$ converges.

Model 4: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3 + 2n}{n^3}$ converges.

Solution. (a–b) Use the Alternating Series test since the series is alternating. (c) Here $a_n = \frac{n^3+2n}{n^3}$. Check the two conditions.

(1) Decreasing? Dividing by n^3, we see $a_n = \frac{n^3+2n}{n^3} = 1 + \frac{2}{n^2}$. These terms decrease as n gets larger:

$$0 \leq 1 + \frac{2}{(n+1)^2} < 1 + \frac{2}{n^2} \quad \text{so} \quad a_{n_1} < a_n.$$

(2) But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3+2n}{n^3} = 1 + \frac{2}{n^2} = 1$.

(e) The Alternating Series Test does not apply. BUT since $\lim_{n\to\infty} a_n \neq 0$, the Divergence Test shows that the series diverges.

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1. Using the approach above, do page 648 #42 and 44.

2. Using the approach above, do page 657 in the following order #16, 14. Optional XC: #18.

3. Optional XC: Determine where the following argument is wrong. Does $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+n}$ converge or diverge?

Solution. (a-b) Looks like $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n}$. Use the Direct Comparison Test. (c) Both series have positive terms. (d) Since $n^2 + n > n$, taking reciprocals changes the direction of the inequality and we have

$$0 < \frac{1}{n^2+n} < \frac{1}{n} \quad \text{so} \quad 0 < \frac{2}{n^2+n} < \frac{2}{n}.$$

(e) Since $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges by the p-series test ($p = 1$), then $\sum_{n=1}^{\infty} \frac{2}{n^2+n}$ diverges by the Direct Comparison Test.

Series Strategy

By the time we finish with series we will have 10 different tests for for convergence and divergence. We need a strategy when we approach a new problem. Here's the method I use:

Start with the easy tests first, which you should be able to do in your head:

(*a*) Does $\lim_{n\to\infty} a_n \neq 0$? If so, the divergence test says the series diverges. Otherwise do more work (most cases).

(*b*) Is it a p-series or geometric series? Does it look like $\sum \frac{1}{k^p}$ or $\sum cr^k$?

1. Next: Are there factorials and/or powers. Try the ratio and root tests.

2. As $k \to \infty$ is it "roughly" a p-series or geometric series? Try the limit comparison or the direct comparison tests.

3. Is it alternating? $\sum(-1)^n a_n$. Try the alternating series test.

4. Can you integrate it? Try the integral test. [Lots of work.]

5. Give your strategy for each series and whether you think it converges.

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3+n}}$

(*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{n!}$

(*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt[5]{n^6}}$

(*d*) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+1)3^n}$

(*e*) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^8+1}{9n^8+n}\right)^{2n}$

(*f*) $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot n!}{n^n}$

(*g*) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^{12}+9n}{7n^{15}+16n^2}$

(*h*) $\displaystyle\sum_{n=1}^{\infty} e\left(\frac{\pi}{e}\right)^{2n}$

(*i*) $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

(*j*) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{1000^{2n}}$

(*k*) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}+\frac{1}{4^n}\right)$

(*l*) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(\frac{2}{n}+\frac{1}{4^n}\right)$

(*m*) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}+\frac{1}{4^n}\right)^{3n}$

(*n*) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

(*o*) $\displaystyle\sum_{n=1}^{\infty} \sec\left(\frac{1}{n^3}\right)$