Math 131 Lab 12: Series

1. Warmup: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$ converges. Give an argument.

2. Determine whether the following series converge. First determine which test to use for each one: Divergence (nth) term, geometric, p-series, ratio, or integral test. Your final answer should consist of a little 'argument' (a sentence or two) and any necessary calculations. **Use appropriate mathematical language**. Here's a **Model Example:**

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ converge?

Preliminary Analysis—Scrap Work: Think about it. Try the easy tests first: Notice that this is not a geometric series or p-series and the Divergence (nth) term test fails ($a_n \to 0$). The ratio test seems inappropriate, no factorials or powers. So we are left with the integral test. Now here's what you might write:

ARGUMENT: Use the integral test. The corresponding function is $f(x) = \frac{1}{(x+1)\ln(x+1)}$ which is positive, decreasing (as x gets bigger, so does the denominator but the numerator stays the same, so the fraction gets smaller), and it is continuous on $[1, \infty)$. The improper integral that we must evaluate is $\displaystyle\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx$. Using a u-substitution with $u = \ln(x+1)$ and $du = \frac{1}{x+1}\,dx$ check that

$$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{(x+1)\ln(x+1)}\,dx = \lim_{b\to\infty} \ln|\ln(x+1)|\Big|_1^b = \lim_{b\to\infty} \ln|\ln(b+1)| - \ln(\ln(2)) = +\infty.$$

Since the integral diverges the integral test says the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ also diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.0101}}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{5\cdot n!}{2^n}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{1+4n^2}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

e) $\displaystyle\sum_{n=1}^{\infty} \ln(2n+3) - \ln(3n+2)$ **f)** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(n+1)!}$ **g)** $\displaystyle\sum_{n=1}^{\infty} \sec\tfrac{1}{n}$ **h)** $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{3}{7}\right)^n$

i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ WeBWorK **j)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3\cdot n!}$ **k)** $\displaystyle\sum_{n=1}^{\infty} 2\arctan(n)$ **l)** $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

m) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ **n)** $\displaystyle\sum_{n=1}^{\infty} 6\left(\frac{5}{2}\right)^n$ **o)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^4-1}{n^4}$ **p)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^6}}$

q) $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n^2+1}$ **r)** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^2+1}$ **s)** $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^2+5n}$ **t)** $4 - \frac{8}{9} + \frac{16}{81} - \frac{32}{729} + \cdots$

3. a) The Divergence (nth) term test says that if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. Does this mean that if $\displaystyle\lim_{n\to\infty} a_n = 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges? Explain your answer. (See the next parts)

b) Give two examples of a series $\sum a_n$ where $\displaystyle\lim_{n\to\infty} a = 0$ and the series diverges.

c) Give two examples of a series $\sum a_n$ where $\displaystyle\lim_{n\to\infty} a = 0$ and the series conerges.

4. Determine whether the following geometric series converge. If so, to what? (Watch the starting indices.)

a) $\displaystyle\sum_{n=2}^{\infty} -4\left(\frac{2}{5}\right)^n$ **b)** $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{-5}{3}\right)^n$ **c)** $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{2^n}{3^{n+3}}\right)$ **d)** $\displaystyle\sum_{n=1}^{\infty} 3\cdot(-2)^n\cdot 7^{-n}$

5. Extra Credit: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converges.

Brief Answers

Full answers will be available online.

1. Ratio Test. Converges.

2. The simplest test to apply... Your answers will include calculations and explanations as in the **ARGUMENT:** on the other side of the page. Ask me to check your work.

a) p-series

b) Ratio Test

c) Integral test

d) p-series test

e) Divergence (nth) term test

f) Ratio Test

g) Divergence (nth) term test

h) Geometric Series Test

i) Integral test

j) Ratio Test

k) Divergence (nth) term test

l) Geometric Series Test

m) Ratio Test

n) Geometric Series Test

o) Divergence (nth) term test

p) p-series test

q) Integral test

r) Divergence (nth) term test

s) Integral test

t) Geometric Series Test

3. a) No. When $\lim_{n\to\infty} a_n = 0$ the the series may converge as it does with the p-series $\sum \frac{1}{n^2}$ where $2 = p > 1$. But it could diverge when $\lim_{n\to\infty} a_n = 0$ the the series may diverge as it does with harmonic series $\sum \frac{1}{n}$ where $1 = p \leq 1$.

4. Remember a geometric series has the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$. Write out the first few terms to determine a and r.

a) $-\frac{16}{15}$.

b) Diverges.

c) $\frac{5}{9}$

d) $-\frac{2}{3}$.

Math 131 Lab 12: Answers

1. ARGUMENT: Factorial: Ratio test. The terms are positive. $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1} = \lim_{n\to\infty} \frac{n+2}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{n+2}{n^2+2n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n}}{1+\frac{2}{n}+\frac{1}{n^2}} = 0$. Since $r < 1$ by the ratio test the series converges.

2. a) ARGUMENT: Converges by the p-series test; $p = 1.0101 > 1$.

b) HW

c) HW

d) ARGUMENT: Diverges by the p-series test; $p = \frac{2}{3} \leq 1$.

e) ARGUMENT: The Divergence (nth) term test.

$$\lim_{n\to\infty} \ln(2n+3) - \ln(3n+2) = \lim_{n\to\infty} \ln\left(\frac{2n+3}{3n+2}\right) = \lim_{n\to\infty} \ln\left(\frac{2+\frac{3}{n}}{3+\frac{2}{n}}\right) = \ln\tfrac{2}{3} \neq 0.$$

By the Divergence (nth) term test the series diverges.

f) ARGUMENT: Factorial: Ratio test. The terms are positive. $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{5^n} = \lim_{n\to\infty} \frac{5}{n+2} 0 < 1$. Since $r < 1$ by the ratio test the series converges.

g) ARGUMENT: Divergence (nth) term test: $\lim_{n\to\infty} \sec\frac{1}{n} = \sec(0) = 1 \neq 0$. By the Divergence ($n$th) term test the series diverges.

h) ARGUMENT: Geometric Series Test: This is a geometric series with $|r| = \frac{3}{7} < 1$. By the geometric series it converges.

i) ARGUMENT: Integral test: Note that $\frac{1}{x\ln x}$ positive and continuous on $[2,\infty)$. It is also decreasing because as x increases, the denominator increases, but the numerator stays the same making the function values smaller. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$. So

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln x|.$$

So

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} n|\ln x|\Big|_2^b = \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2| = \infty.$$

Since $\int_2^\infty \frac{1}{x\ln x}\,dx$ diverges so does $\sum_{n=2}^\infty \frac{1}{n\ln n}$ by the integral test.

j) ARGUMENT: Factorial: Ratio test. The terms are positive. $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3\cdot(n+1)!} \cdot \frac{3\cdot n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$. Since $r > 1$ by the ratio test the series diverges.

k) HW

l) ARGUMENT: Geometric Series Test: Here $|r| = |-1| = 1$. Diverges by the geometric series test. Or use the Divergence (nth) term test: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$ DNE $\neq 0$. So by the Divergence test the series diverges.

m) ARGUMENT: Powers: Ratio test. The terms are positive. $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)e^{-n-1}}{ne^{-n}} = \lim_{n\to\infty} \frac{(n+1)e^{-1}}{n} = e^{-1} < 1$. Since $r < 1$ by the ratio test the series converges. (This is actually easier to do by the root test, which we will cover next.)

n) ARGUMENT: Geometric Series Test: This is a geometric series with $|r| = \frac{5}{2} > 1$. So it diverges. (Or use the nth term test.)

o) ARGUMENT: Divergence (nth) term test. $\lim_{n\to\infty} \frac{n^4-1}{n^4} = \lim_{n\to\infty} 1 - \frac{1}{n^4} = 1 \neq 0$. By the Divergence ($n$th) term test the series diverges.

p) HW

q) ARGUMENT: Integral test: Note that $\frac{2x}{x^2+1}$ positive, continuous, and decreasing since $f'(x) = \frac{1-2x^2}{(x^2+1)^2} < 0$ on $[1, \infty)$. Let $u = x^2 + 1$. Then $du = 2x\, dx$. So

$$\int_1^\infty \frac{2x}{x^2+1}\, dx = \lim_{b\to\infty} \int_b^\infty \frac{2x}{x^2+1}\, du = \lim_{b\to\infty} \ln|x^2+1|\Big|_1^b = \lim_{b\to\infty} \ln|b^2+1| - \ln 1 = \infty.$$

Since $\int_1^\infty \frac{2x}{x^2+1}\, dx$ diverges, so does $\sum_{n=1}^\infty \frac{2n}{n^2+1}$ by the integral test.

r) ARGUMENT: Divergence (nth) term test. Remember if $f(x) = a^x$, then $f'(x) = (\ln a)x^x$. So

$$\lim_{n\to\infty} \frac{3^n}{n^2+1} = \lim_{x\to\infty} \frac{3^x}{x^2+1} = \lim_{x\to\infty} \frac{(\ln 3)3^x}{2x} = \lim_{x\to\infty} \frac{(\ln 3)^2 3^x}{2} = \infty \neq 0.$$

By the Divergence (nth) term test the series diverges.

s) ARGUMENT: The integral test. Note that $f(x) = \frac{10}{x^2+5x}$ is positive and continuous on $[1, \infty)$. It is also decreasing because as x increases, the denominator increases, but the numerator stays the same making the function values smaller. Or use $f'(x) = -10(2x+5)(x^2+5x)^{-2} < 0$ on $[1, -\infty)$. Use partial fractions.

$$\int_1^\infty \frac{10}{x^2+5x}\, dx = \lim_{b\to\infty} \int_1^b \frac{2}{x} - \frac{2}{x+5}\, dx = \lim_{b\to\infty} 2\ln|x| - 2\ln|x+5|\Big|_1^b = \lim_{b\to\infty} 2\ln\frac{x}{x+5}\Big|_1^b$$

$$= \lim_{b\to\infty} 2\ln\frac{b}{b+5} - 2\ln\frac{1}{6} = \lim_{b\to\infty} 2\ln\frac{1}{1+\frac{5}{b}} - 2\ln\frac{1}{6} = 2\ln 1 - 2\ln\frac{1}{6} = \ln 36.$$

Since the integral converges, so does the corresponding series $\sum_{n=1}^\infty \frac{10}{n^2+5n}$ by the integral test.

t) HW

3. No. For example the series $\sum \frac{1}{n}$ diverges by the p-series test. But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So even though the Divergence (nth) term $\to 0$, the series still diverges.

4. Remember a geometric series has the form $\displaystyle\sum_{n=0}^\infty ar^n = a + ar + ar^2 + ar^3 + \cdots$. Write out the first few terms to determine a and r.

a) $\displaystyle\sum_{n=2}^\infty -4\left(\frac{2}{5}\right)^n = -\underbrace{\frac{16}{25}}_{a} - \underbrace{\frac{32}{125}}_{ar} - \underbrace{\frac{64}{625}}_{ar^2} - \cdots$. $a = -\frac{16}{25}$, $r = \frac{2}{5}$. Sum: $\dfrac{-\frac{16}{25}}{1-\frac{2}{5}} = -\dfrac{16}{15}$.

b) Diverges since $|r| = \frac{5}{3} > 1$.

c) $\displaystyle\sum_{n=0}^\infty 5\frac{2^n}{3^{n+3}} = \underbrace{\frac{5}{27}}_{a} + \underbrace{\frac{10}{81}}_{ar} + \underbrace{\frac{20}{243}}_{ar^2} + \cdots$. $a = \frac{5}{27}$, $r = \dfrac{a_{n+1}}{a_n} = \dfrac{\frac{10}{81}}{\frac{5}{27}} = \dfrac{2}{3}$. Sum: $\dfrac{\frac{5}{27}}{1-\frac{2}{3}} = \dfrac{5}{9}$

d) $\displaystyle\sum_{n=1}^\infty 3\cdot(-2)^n\cdot 7^{-n} = -\underbrace{\frac{6}{7}}_{a} + \underbrace{\frac{12}{49}}_{ar} - \underbrace{\frac{24}{343}}_{ar^2} + \cdots$. $a = -\frac{6}{7}$, $r = \dfrac{a_{n+1}}{a_n} = \dfrac{\frac{12}{49}}{-\frac{6}{7}} = -\dfrac{2}{7}$. Sum: $\dfrac{-\frac{6}{7}}{1+\frac{2}{7}} = -\dfrac{2}{3}$.

5. Use the integral test with triangles. $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, and $\sqrt{x^2+1} = \sec\theta$. So

$$\int \frac{1}{\sqrt{x+1}}\, dx = \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c = \ln|\sqrt{x^2+1} + x| + c.$$

So

$$\int_1^\infty \frac{1}{\sqrt{x+1}}\, dx = \lim_{b\to\infty} \ln|\sqrt{x^2+1} + x|\Big|_1^b = \lim_{b\to\infty} \ln|\sqrt{b^2+1} + b| - \ln|\sqrt{2}+1| = \infty. \quad \text{Diverges.}$$

Since the integral diverges, so does the corresponding series by the integral test.