Math 131, Lab 13

Make sure to get your answers checked by a TA or me as you go along.

1. Go through this list of series and use a strategy to decide which test is appropriate to determine whether the series converges. In most cases you should be able to "predict" or outline whether the series converges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+4n}}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{(2n)!}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{(n+1)2^n}$

d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n^3+1}{7n^3+n}\right)^{2n}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot n!}{n^n}$

f) $\displaystyle\sum_{n=1}^{\infty} \pi\left(\frac{e}{\pi}\right)^{\pi n}$

g) $\displaystyle\sum_{n=1}^{\infty} \tan\frac{1}{n^3}$

h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n\cdot 1000^{2n}}$

i) $\displaystyle\sum_{n=1}^{\infty} \frac{12n^{34}+9n^9}{10n^{35}+n+1}$

j) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}+\frac{1}{2^n}\right)$

k) $\displaystyle\sum_{n=1}^{\infty} [\ln(3n+5)-\ln(2n+4)]$

l) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n}+\frac{1}{2^n}\right)$

m) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}+\frac{1}{2^n}\right)^{3n}$

n) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{2n^2}$

2. Determine whether these alternating series converge. **Justify your answers with a complete argument.**

a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\tfrac{1}{n}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{3n+2}$ **c)** $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{3(-1)^n}{n^2+5n+4}$

3. How return to Problem 1 and do these parts in the order given: (a), (j), (i), (g), (m), (b), (k). **Justify your answers with a complete argument.**

4. Determine whether these series converge. **Justify your answers with a complete argument.**

a) Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6+1}}$ converge?

b) Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n^6+1}}$ converge?

1. Short Answers

a) LC **b)** Rat **c)** LC/Rat **d)** Roo **e)** Rat **f)** Geo/Roo/Rat
g) LC **h)** Roo/Rat **i)** LC **j)** DC **k)** Div **l)** AST
m) Roo/Rat **n)** Roo

2. Short Answers

a) Co **b)** Di **c)** Co **d)** Co

3. Short Answers

a) Di **j)** Di **i)** Di **n)** Co **b)** Co **k)** Di

Lab 13 Answers

1. a) (a) Apply the Limit Comparison. (b) Similar to $\sum \frac{1}{n^{2/3}}$. (c) $\frac{1}{\sqrt[3]{n^2+4n}}$ and $\frac{1}{n^{2/3}}$ are always positive. (d)

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+4n}} \cdot \frac{n^{2/3}}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^{2/3}}{\sqrt[3]{n^2}} = 1 > 0.$$

(e) Since $0 < L = 1 < \infty$ and since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges by the p-series test ($p = \frac{2}{3} \le 1$), then $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+4n}}$ diverges by the limit comparison test.

b) (a) Ratio test. (b) Factorials. (c) The terms $\frac{2^{n+1}}{(2n)!}$ are positive. (d)

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+2}}{(2n+2)!} \cdot \frac{(2n)!}{2^{n+1}} = \lim_{n\to\infty} \frac{2}{(2n+1)(2n+2)} = 0 < 1.$$

(e) Since $r < 1$, by the ratio test the series converges.

c) (a) Apply the Limit Comparison test. (b) Notice (HP) $\sum \frac{3n}{(n+1)2^n} \approx \sum \frac{3}{2^n}$. So compare to the geometric series $\sum \frac{1}{2^n}$. (c) $\frac{3n}{(n+1)2^n}$ and $\frac{1}{2^n}$ are both positive. (d) So

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n}{(n+1)2^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{3n}{n+1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{3n}{n} = 3 > 0.$$

(e) Since $0 < L = 3 < \infty$ and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by the geometric series test ($|r| = \frac{1}{2} < 1$), then $\sum_{n=1}^{\infty} \frac{3n}{(n+1)2^n}$ converges by the limit comparison test. OR: (a-b) Ratio test, because there are nth powers. (c) The terms are positive. (d)

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3(n+1)}{(n+2)2^{n+1}} \cdot \frac{(n+1)2^n}{3n} = \lim_{n\to\infty} \frac{(n+1)}{(n+2)2} \cdot \frac{(n+1)}{n} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2+2n} \overset{\text{HP}}{=} \frac{1}{2} < 1.$$

(e) Since $r < 1$ by the ratio test the series converges.

d) (a) Root test (b) because there are powers. (c) The terms $\left(\frac{4n^3+1}{7n^3+n}\right)^{2n}$ are positive. (d)

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{4n^3+1}{7n^3+n}\right)^{2n}\right]^{1/n} = \lim_{n\to\infty} \left(\frac{4n^3+1}{7n^3+n}\right)^2 \overset{\text{HP}}{=} \left(\frac{4}{7}\right)^2 < 1.$$

(e) Since $r < 1$ by the root test the series converges.

e) (a) Ratio test (b) because of the factorial. (c) The terms are positive. (d) $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{2 \cdot (n+1)!} \cdot \frac{2 \cdot n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right)^n = \frac{e}{1} > 1$. (e) Since $r > 1$ by the ratio test the series diverges.

f) (a) Geometric series test. (b) We can rewrite it as the geometric series $\sum_{n=1}^{\infty} \pi \left[\left(\frac{e}{\pi}\right)^{\pi}\right]^n$. (d) $|r| = (\frac{e}{\pi})^{\pi} < 1$ Since $|r| < 1$, the geometric series test says the series converges.

g) (a) Limit comparison test. (b) Looks like the p-series $\sum \frac{1}{n^3}$. (c) The terms are positive because $0 < \frac{1}{n^3} < 1 < \pi/2$, so $\tan \frac{1}{n^3} > 0$.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan \frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{x\to\infty} \frac{\tan \frac{1}{x^3}}{\frac{1}{x^3}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x^3})(-\frac{3}{x^4})}{-\frac{3}{x^4}} = \lim_{x\to\infty} \sec^2 \frac{1}{x^3} = \sec^2 0 = 1.$$

Since $0 < L = 1 < \infty$ and since $\sum \frac{1}{n^3}$ converges by the p-series test ($p = 3 > 1$), then $\sum \tan \frac{1}{n^3}$ converges by the limit comparison test.

h) (a–b) Use the root test because of the powers. (c) The terms $\frac{n^n}{2^n \cdot 1000^{2n}}$ are positive. (d)

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\frac{n^n}{2^n \cdot 1000^{2n}}\right]^{1/n} = \lim_{n\to\infty} \left(\frac{n}{2 \cdot 1000^2}\right) = \infty > 1.$$

(e) Since $r > 1$ by the root test the series diverges.

i) (a) Limit comparison test. (b) Looks like the p-series $\sum \frac{1}{n}$. (c) The terms of both series are clearly positive.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{12n^{34} + 9n^9}{10n^{35} + n + 1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{12n^{35} + 9n^{10}}{10n^{35} + n + 1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{12n^{35}}{10n^{35}} = \frac{12}{10}.$$

Since $0 < L = \frac{12}{10} < \infty$ and since $\sum \frac{1}{n}$ diverges by the p-series test ($p = 1 \le 1$), then $\sum_{n=1}^{\infty} \frac{12n^{34}+9n^9}{10n^{35}+n+1}$ diverges by the limit comparison test.

j) (a) Direct comparison with $\sum \frac{1}{n}$. (b) Notice that the terms are larger than those in $\sum \frac{1}{n}$, so we expect the series to diverge. (c–d) Specifically $0 < \frac{1}{n} < \frac{1}{n} + \frac{1}{2^n}$. (e) Since $\sum \frac{1}{n}$ diverges (p-series, $p = 1$), then by direct comparison, the LARGER series $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right)$ also converges.

k) (a–c) Divergence Test: the nth term does not go to 0. (d) $\lim_{n\to\infty} [\ln(3n + 5) - \ln(2n + 4)] = \lim_{n\to\infty} \ln\left(\frac{3n+5}{2n+4}\right) \overset{\text{HP}}{=} \ln\left(\frac{3n}{2n}\right) = \ln \frac{3}{2} \neq 0$. (e) By the Divergence Test, the series diverges.

l) (a–c) Use the alternating series test with $a_n = \frac{1}{n} + \frac{1}{2^n} > 0$ (positive terms). (d) Check the two conditions: (i): $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{2^n} \overset{\text{KeyLim}}{=} 0 + 0 = 0.$✓ (ii): Decreasing? As n increases, the numerators are constant and the denominators increase, so the terms are decreasing. [Or take the derivative $f(x) = \frac{1}{x} + (\frac{1}{2})^x = x^{-1} + (\frac{1}{2})^x$ so $f'(x) = -x^{-2} + \ln(\frac{1}{2})(\frac{1}{2})^x < 0$✓ $(x \ge 1)$ (Remember $\ln(\frac{1}{2}) < 0$.) So $f(x)$ and a_n are decreasing.] (e) By the Alternating Series Test the series converges.

m) (a–b) Root test because of the powers. (c) The terms are positive $\left(\frac{1}{n} + \frac{1}{2^n}\right)^{3n}$ are positive. (d) So

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{1}{n} + \frac{1}{2^n}\right)^{3n}\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right)^3 \overset{\text{KeyLim}}{=} (0 + 0)^3 = 0 < 1.$$

(e) Since $r < 1$ by the root test the series converges.

n) (a–b) Root test because of the powers. (c) The terms are positive $\left(\frac{n}{n+1}\right)^{2n^2}$ are positive. (d) So

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{2n^2}\right]^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} = \lim_{n\to\infty} \left[\left(\frac{1}{1+1/n}\right)^n\right]^2 \overset{\text{Keylim}}{=} \left(\frac{1}{e}\right)^2 < 1$$

(e) Since $r < 1$ by the root test the series converges.

2. a) (a) The series is alternating. (b–c) Use the alternating series test with $a_n = \frac{1}{\sqrt[5]{n^2}}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[5]{n^2}} = 0.$✓

2. Decreasing? Is $a_{n+1} \le a_n$? Because $\sqrt[5]{(n+1)^2} \ge \sqrt[5]{n^2}$, so taking reciprocals changes the inequality to $\frac{1}{\sqrt[5]{(n+1)^2}} \le \frac{1}{\sqrt[5]{n^2}}$. The sequence is decreasing. OR: if $f(x) = \frac{1}{\sqrt[5]{(x^2}} = x^{-2/5}$ for $x \ge 1$, then $f'(x) = -\frac{2}{5}x^{-7/5} < 0$, so the function and the sequence are decreasing.✓ (e) Since the series satisfies the two conditions, by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[5]{n^2}}$ converges.

b) (a–c) Use the alternating series test with $a_n = \frac{2n+1}{3n+2}$. (d) Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{3n+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n}{3n} = \frac{2}{3} \neq 0$. Fails. (e) Since this hypothesis is not satisfied, the alternating series test does not apply. However, by the nth term test the series diverges since the nth term does not go to 0.

c) (a–c) Use the alternating series test with $a_n = \frac{\ln n}{n}$. (d) Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$✓

2. Decreasing? Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0$ for $x \geq 3$. The sequence is decreasing.✓(e) Since the series satisfies the two hypotheses, by the Alternating Series test, $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges.

d) (a–c) Use the alternating series test with $a_n = \frac{3}{n^2+5n+4}$. (d) Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3}{n^2+5n+4} = 0.$✓

2. Decreasing? As n increases, the numerator is constant and the denominator increases. So the sequence decreases. (Or use $f'(x) = \frac{-3(2x+5)}{(x^2+5x+4)^2} < 0$ for $x \geq 1$. The sequence is decreasing.✓

(e) Since the series satisfies the two hypotheses, by the Alternating Series test, so $\displaystyle\sum_{n=1}^{\infty} \frac{3(-1)^n}{n^2+5n+4}$ converges.

$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6+1}}$

3. See #1

4. a) (a) Limit comparison test. (b) $\sum \frac{n^2}{\sqrt{n^6+1}} \approx \sum \frac{n^2}{\sqrt{n^6}} = \sum \frac{n^2}{n^3} = \sum \frac{1}{n}$ (c) The terms are positive.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^6+1}} \cdot \frac{n}{1} \overset{\text{HP}}{=} \lim_{n\to\infty} \frac{n3}{\sqrt{n^6}} = \lim_{n\to\infty} \frac{n3}{n^3} = 1.$$

Since $0 < L = 1 < \infty$ and since $\sum \frac{1}{n}$ diverges by the p-series test ($p = 1 \leq 1$), then $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6+1}}$ diverges by the limit comparison test.

b) (a–c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n^6+1}}$. Use the alternating series test with $a_n = \frac{n^2}{\sqrt{n^6+1}}$. (d) Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2}{\sqrt{n^6+1}} \overset{\text{HP}}{=} \lim_{n\to\infty} \dfrac{n^2}{\sqrt{n^6}} \lim_{n\to\infty} \dfrac{n^2}{n^3} = \lim_{n\to\infty} \dfrac{1}{n} = 0.$✓

2. Decreasing? Let $f(x) = \frac{x^2}{\sqrt{x^6+1}}$

$$f'(x) = \frac{2x\sqrt{x^6-1} - \frac{x^2 \cdot 6x^5}{2\sqrt{x^6+1}}}{x^6+1} = \frac{2x\sqrt{x^6-1} - \frac{3x^7}{\sqrt{x^6+1}}}{x^6+1} = \frac{\frac{2x(x^6-1)-3x^7}{\sqrt{x^6+1}}}{x^6+1} = \frac{\frac{2x-x^7}{\sqrt{x^6+1}}}{x^6+1} < 0$$

for $x > 1$. The sequence is decreasing.✓

(e) Since the series satisfies the two hypotheses, by the Alternating Series test, so does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n^6+1}}$. converges.