Math 131 Lab 14: Series

1. Determine whether these arguments are correct. If not, correct them.

a) Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n 2n^4}{6n^9-1}$ converges absolutely, conditionally, or diverges. ARGUMENT: Use the alternating series test with $a_n = \frac{2n^4}{6n^9-1} > 0$. Check the two conditions: (i): $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^4}{6n^9-1} = \lim_{n\to\infty} \frac{2}{6n^5} = 0.$✓(ii): Decreasing? Take the derivative! (You can't just say the bottom gets bigger since the top gets bigger, too!) $f(x) = \frac{2x^4}{6x^9-1}$ so

$$f'(x) = \frac{8x^3(6x^9-1) - (2x^4)54x^8}{(6x^9-1)^2} = \frac{-60x^{12} - 8x^3}{(6x^9-1)^2} < 0$$

so $f(x)$ and a_n are decreasing.✓So by the Alternating Series Test the series converges conditionally.

b) Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{6n+2}$ converges absolutely, conditionally, or diverges. ARGUMENT: Notice $\sum_{n=1}^{\infty} \left|\frac{(-1)^n(2n+1)}{6n+2}\right| = \sum_{n=1}^{\infty} \frac{2n+1}{6n+2} = \lim_{n\to\infty} \frac{2n+1}{6n+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n}{6n} = \frac{1}{3}$. So the series converges absolutely.

c) Determine the radius and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{x^n}{3n^2+1}$. ARGUMENT: We know that the series converges at its center $a = 0$. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{3(n+1)^2+1} \cdot \frac{3n^2+1}{x^n}\right| = \lim_{n\to\infty} \left|\frac{3n^2+1}{3n^2+6n+4} \cdot x\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{3n^2}{3n^2} \cdot x\right| = |x|.$$

By the ratio test, the series *converges* if $|x| < 1$ and *diverges* when $|x| > 1$. The radius of convergence is $R = 1$. Check the endpoints $a - R = 0 - 1 = -1$ and $a + R = 0 + 1 = 1$. **For** $x = 1$**:** We get $\sum_{n=0}^{\infty} \frac{1}{3n^2+1}$. Since $0 < \frac{1}{3n^2+1} < \frac{1}{n^2}$ and since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), then by direct comparison $\sum_{n=0}^{\infty} \frac{1}{3n^2+1}$ converges. **For** $x = -1$**:** We get $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^2+1}$. However, we just saw that $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{3n^2+1}\right| = \sum_{n=0}^{\infty} \frac{1}{3n^2+1}$ converges. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^2+1}$ converges by the absolute convergence test. The interval of convergence is $[-1, 1]$ and includes both endpoints.

2. Determine whether these series converge absolutely, conditionally, or not at all. (Hint: Remember to use theRatio Test Extension for absolute convergence/divergence, when it is appropriate.)

a) $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{3n^2+2}$ **b)** $\sum_{n=1}^{\infty} \frac{(-1)^n n 2^n}{3^{n+1}}$ **c)** $\sum_{n=1}^{\infty} \frac{n!}{(-3)^n}$

3. Find the radius R and interval of convergence for each of these series. Remember the endpoints.

a) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$ **b)** $\sum_{n=1}^{\infty} \frac{(-1)^n n! x^n}{4n}$ **c)** $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$

d) $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n+1}$ **e)** $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{3^{2n}}$ **f)** $\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$

4. EXTRA FUN: If you finish early: Find the radius of convergence R for $\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$ and for $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$.

Brief Answers

1. a) Wrong. What didn't the student check? (b) Wrong, in so many ways. (c) Correct. This is a model of how I want your answers in #3 to be!

2. Conditional, Absolute, Diverge.

3. Lots of justification (words) required!

a) $[-1/2, 1/2]$ **b)** $\{0\}$ **c)** $(-3, 3)$ **d)** $[3, 5)$ **e)** $(-8, 10)$ **f)** $(-\infty, \infty)$

4. Ask.

Background

1. Power Series. For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at a, precisely one of the following is true.

a) The series converges only at $x = a$.

b) There is a real number $R > 0$ so that the series converges absolutely for $|x - a| < R$ and diverges for $|x - a| > R$.

c) The series converges for all x.

NOTE: In case (b) the power series may converge at both endpoints $a - R$ and $a + R$, either endpoint, or neither endpoint. You must check the convergence at the endpoints separately. Here's what the intervals of convergence can look like:

$[a-r, a+r]$ $[a-r, a+r)$ $(a-r, a+r]$ $(a-r, a+r)$

2. The Ratio Test Extension. Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **non-zero** terms and let $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

1. If $r < 1$, then the series $\sum a_n$ converges *absolutely.*
2. If $r > 1$ (including ∞), then the series $\sum a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

3. The Alternating Series Test. Assume $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the follow two conditions hold:

a) $\lim_{n\to\infty} a_n = 0$

b) $a_{n+1} \leq a_n$ for all n (i.e., a_n is decreasing which can also be tested by showing $f'(x) < 0$).

4. Absolute Convergence Test. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

Lab 14 Answers

1. a) Wrong. The student never checked for absolute convergence. ARGUMENT: First we check absolute convergence. $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n^4)}{6n^9-1}\right| = \sum_{n=1}^{\infty}\frac{2n^4}{6n^9-1}$. Notice that $\frac{2n^4+1}{6n^9+2}\approx\frac{1}{n^5}$. So let's use the limit comparison test. The terms of the series are positive and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^4}{6n^9-1}\cdot\frac{n^5}{1} = \lim_{n\to\infty}\frac{2n^9}{6n^9-1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n^9}{6n^9} = \frac{1}{3}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges (p-series with $p=5>1$), then $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n^4)}{6n^9-1}\right|$ converges by the limit comparison test. So the series converges absolutely.

b) Wrong. Puleeze never do this! You might start with the alternating series test with $a_n = \frac{2n+1}{6n+2}\neq 0$. Check the two conditions (i): $\displaystyle\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{2n+1}{6n+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n}{6n} = \frac{1}{3}$. So the series diverges by the nth term test (not the alternating series test). It's the nth term test.

c) Correct. This is what I want your answers in #3 to be!

2. a) $\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{3n^2+2}$. The series is similar to $\sum_{n=1}^{\infty}\frac{1}{n}$, so it probably will not converge absolutely. ARGUMENT: Use the alternating series test with $a_n = \frac{2n+1}{3n^2+2} > 0$. Check the two conditions: (i): $\displaystyle\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{2n+1}{3n^2+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n}{3n^2} = \lim_{n\to\infty}\frac{2}{3n} = 0.$✓ (ii): Decreasing? Take the derivative. (You can't just say the bottom gets bigger since the top gets bigger, too!) $f(x) = \frac{2x+1}{3x^2+2}$ so

$$f'(x) = \frac{2(3x^2+2)-(2x+1)6x}{(3x^2+2)^2} = \frac{-6x^2-6x^2+4}{(3x^2+2)^2} < 0 ✓ \qquad (x\geq 1)$$

so $f(x)$ and a_n are decreasing. By the Alternating Series Test the series converges. Now check absolute convergence. $\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n+1)}{3n^2+2}\right| = \sum_{n=1}^{\infty}\frac{2n+1}{3n^2+2}$. Compare to $\sum_{n=1}^{\infty}\frac{1}{n}$. Notice both $\frac{1}{n}>0$ and $\frac{2n+1}{3n^2+2}>0$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+1}{3n^2+2}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{2n^2+n}{3n^2+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n^2}{3n^2} = \frac{2}{3} > 0.$$

Since $\sum_{n=0}^{\infty}\frac{1}{n}$ diverges by the p-series test ($p=1$), then by the limit comparison test $\sum_{n=0}^{\infty}\frac{2n+1}{3n^2+1}$ also diverges. So the series is conditionally convergent.

b) $\sum_{n=1}^{\infty}\frac{(-1)^n n2^n}{3^{n+1}}$. Because of the nth powers, try the ratio test extension first (testing for absolute convergence) with $a_n = \frac{n2^n}{3^{n+1}}\neq 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)2^{n+1}}{3^{n+2}}\cdot\frac{3^{n+1}}{(-1)^n n2^n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{3n}\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n}{3n} = \frac{2}{3} < 1.$$

By the ratio test the series converges absolutely.

c) $\sum_{n=1}^{\infty}\frac{n!}{(-3)^n}$. Because of the nth power and factorial, try the ratio test first (testing for absolute convergence) with $a_n = \frac{n!}{(-3)^n}\neq 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(-3)^{n+1}}\cdot\frac{(-3)^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \infty > 1.$$

By the ratio test the series diverges.

3. a) $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2}$. ARGUMENT: The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(2x)^n}\right| = \lim_{n\to\infty}\left|\frac{n^2 2x}{(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{n^2}{n^2+2n+1}\cdot 2x\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\left|\frac{n^2}{n^2}\cdot 2x\right| = 2|x|.$$

By the ratio test, the series *converges* if $2|x| < 1 \iff |x| < \frac{1}{2}$ and *diverges* when $|x| > \frac{1}{2}$. The radius of convergence is $R = \frac{1}{2}$. Check the endpoints $-\frac{1}{2}$ and $\frac{1}{2}$? **For** $x = \frac{1}{2}$**:** We get

$$\sum_{n=0}^{\infty}\frac{(2(\frac{1}{2}))^n}{n^2} = \sum_{n=0}^{\infty}\frac{1}{n^2}$$

which converges by the p-series test ($p = 2 > 1$). **For** $x = -\frac{1}{2}$**:** We get

$$\sum_{n=0}^{\infty}\frac{(2(-\frac{1}{2}))^n}{n^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$$

But this series converges absolutely (we just did it). The interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n! x^n}{4n}$. ARGUMENT: The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!x^{n+1}}{4(n+1)}\cdot\frac{4n}{(-1)^n n! x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)n}{n+1}\cdot x\right| = \lim_{n\to\infty}|nx| = \infty > 1.$$

By the ratio test, the series *diverges* when $|x| > 0$. The radius of convergence is $R = 0$. The series only converges at $x = 0$.

c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}$. The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{9^{n+1}}\cdot\frac{9^n}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{9}\right|.$$

By the ratio test, the series *converges* if $|\frac{x^2}{9}| < 1 \iff |x^2| < 9 \iff |x| < 3$ and *diverges* when $|x| > 3$. The radius of convergence is $R = 3$. Check the endpoints $a - R = 0 - 3 = -3$ and $a + R = 0 + 3 = 3$. **For** $x = 3$**:** We get

$$\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n$$

which diverges by the geometric series test ($|r| = 1$). The same is true **for** $x = -3$**:** Again we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n.$$

The interval of convergence is $(-3, 3)$.

d) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-4)^n}{n+1}$. ARGUMENT: The series converges at its center $a = 4$. Apply the ratio test. For any $x \neq 4$

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{n+2}\cdot\frac{n+1}{(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+1}\cdot(x-4)\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\left|\frac{n}{n}\cdot(x-4)\right| = |x-4|.$$

By the ratio test, the series *converges* if $|x - 4| < 1$ and *diverges* when $|x - 4| > 1$. The radius of convergence is $R = 1$. Check the endpoints: $a - R = 4 - 1 = 3$ and $a + R = 4 + 1 = 5$. **For** $x = 3$**:** We get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$$

Use the alternating series test with $a_n = \frac{1}{n+1} \neq 0$. Check the two conditions. (1) $\lim_{n\to\infty} \frac{1}{n+1} = 0$✓ and (2) $a_{n+1} \leq a_n$ since $\frac{1}{n+2} < \frac{1}{n+1}$ (decreasing).✓ So the series converges at $x = 3$ by the alternating series test. **For $x = 5$:** We get

$$\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

Use the limit comparison test with $\sum_{n=0}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} 1.$$

Since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges by the p-series test ($p = 1$), then by the limit comparison test $\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges. The interval of convergence is $[3, 5)$.

e) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-1)^n}{3^{2n}}$. The series converges at its center $a = 1$. Apply the ratio test. For any $x \neq 1$

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-1)^{n+1}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{n(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{3^2 n} \cdot (x-1) \right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left| \frac{n}{9n} \cdot (x-1) \right| = \left| \frac{x-1}{9} \right|.$$

By the ratio test, the series *converges* if $\left|\frac{(x-1)}{9}\right| < 1 \iff |x-1| < 9$ and *diverges* when $|x-1| > 9$. The radius of convergence is $R = 9$. What about the endpoints $a - R = 1 - 9 = -8$ and $a + R = 1 + 9 = 10$? **For $x = -8$:** We get

$$\sum_{n=1}^{\infty} \frac{n(1+8)^{2n}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{n(-9)^n}{9^n} = \sum_{n=1}^{\infty} (-1)^n n.$$

But $\lim_{n\to\infty} (-1)^n n$ DNE, so by the nth term test the series diverges. **For $x = 10$:** We get

$$\sum_{n=1}^{\infty} \frac{n(10-1)^n}{9^n} = \sum_{n=1}^{\infty} n$$

and so the series diverges by the nth term test again. The interval of convergence is $(-8, 10)$.

f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$. The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{(2n+1)(2n+2)} \cdot x \right| = \lim_{n\to\infty} \left| \frac{1}{4n+2} \cdot x \right| = 0 < 1.$$

By the ratio test, the series for all x The interval of convergence is $(-\infty, \infty)$.

4. a) $\sum_{n=1}^{\infty} \frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n+1)}$. Use the ratio test.

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{1\cdot 3\cdot 5\cdots(2n+3)} \cdot \frac{1\cdot 3\cdot 5\cdots(2n+1)}{n!x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{2n+3} \right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left| \frac{nx}{2n} \right| = \lim_{n\to\infty} \left| \frac{x}{2} \right|.$$

By the ratio test, the series *converges* if $|\frac{x}{2}| < 1 \iff |x| < 2$ and *diverges* when $|x| > 2$. The radius of convergence is $R = 2$.

b) $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$ Use the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n x^n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}x}{(n+1)n^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)^n x}{n^n} \right| \\ &= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^n \cdot x \right| = \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right)^n \cdot x \right| = |ex|. \end{aligned}$$

By the ratio test, the series *converges* if $|ex| < 1 \iff |x| < \frac{1}{e}$. The radius of convergence is $R = \frac{1}{e}$.