1. (22 points) Easy pieces to get you started. (2 points each, unless noted.)

a) Determine $\lim_{n\to\infty}\left(1-\frac{4}{n}\right)^{2n}$. $= \lim_{n\to\infty} [(1-4/n)^n]^2 = (e^{-4})^2$

e^{-8}

Answer

b) Determine $\int_1^\infty \frac{4}{x^{44}}\,dx$.

$\frac{4}{43}$

Answer

c) Determine whether $\sum_{n=1}^{\infty} n^{-4}$ converges. Justify your answer with a brief argument.

$= \sum \frac{1}{n^4}$ p-series $p=4>1$

converges

Answer

d) Determine $\int \cos^2(4x)\,dx$. $= \int \frac{1}{2}+\frac{1}{2}\cos 8x\,dx$

$\frac{1}{2}x + \frac{1}{16}\sin 8x + C$

Answer

e) Determine $\int \sec 4x\,dx$.

$\frac{1}{4}\ln|\sec 4x + \tan 4x| + C$

Answer

f) Determine $\int 4^x + \frac{1}{4\sqrt[4]{x^3}}\,dx$. $= \int 4^x + \frac{1}{4}x^{-3/4}\,dx$

$\frac{4^x}{\ln 4} + x^{1/4} + C$

Answer

g) Determine $\int x^3\cos(x^4+4)\,dx$.

$\frac{1}{4}\sin(x^4+4) + C$

Answer

h) Consider the function $f(x) = \frac{1}{x^2}$ on $[1,4]$ Is the left-hand Riemann sum Left(n) an overestimate or underestimate of $\int_1^4 \frac{1}{x^2}\,dx$? Explain in one sentence.

$f'(x) = -2x^{-3} < 0$ ... $f(x)$ is decreasing

Overestimate

i) Find the sum of the series $\sum_{n=2}^{\infty} \frac{4}{\sqrt{k+1}} - \frac{4}{\sqrt{k}}$, or explain why it does not converge.

$S_n = \left(\cancel{\frac{4}{\sqrt{3}}} - \frac{4}{\sqrt{2}}\right) + \left(\cancel{\frac{4}{\sqrt{4}}} - \cancel{\frac{4}{\sqrt{3}}}\right) + \dots + \left(\frac{4}{\sqrt{n+1}} - \cancel{\frac{4}{\sqrt{n}}}\right)$

$S_n = \frac{4}{\sqrt{n+1}} - \frac{4}{\sqrt{2}}$

$\lim_{n\to\infty} S_n = -\frac{4}{\sqrt{2}}$

Answer

j) (4 points) Find the sum of the series $\sum_{n=2}^{\infty} 3\left(-\frac{4}{5}\right)^n$, or explain why it does not converge.

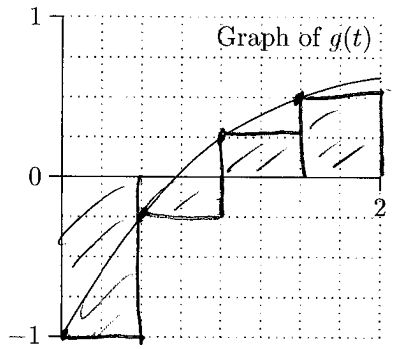
r

$= \underbrace{\frac{48}{25}}_{a} - \frac{192}{125}$

$\frac{a}{1-r} = \frac{\frac{48}{25}}{1+\frac{4}{5}} = \frac{\overset{16}{\cancel{48}}}{\underset{5}{\cancel{25}}}\cdot\frac{\cancel{5}}{\underset{3}{\cancel{9}}} = \frac{16}{15}$

Answer

2. **a)** (4 points) Draw and then estimate the **left-hand** Riemann sum Left(4) for the graph of g on $[0, 2]$ below.



$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

$\text{Left}(4) = \frac{1}{2}(-1 - 1/4 + 1/4 + 1/2)$

$= \frac{1}{2}(-1/2)$

$= -1/4$

$-1/4$

Answer

b) (2 points) If $g(t)$ is graphed above and $G(x) = \int_0^x g(t)\,dt$, on what interval is $G(x)$ decreasing? $(0, 3/4)$

c) (8 points) Let $f(x) = 1 + 4x^2$ on the interval $[0, 3]$. Find and simplify the expression for the right-hand endpoint Riemann sum Right(n). Finally, evaluate $\lim_{n\to\infty} \text{Right}(n)$. How can you **check your answer**?

$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = \frac{3i}{n}$

$f(x_i) = 1 + 4\left(\frac{3i}{n}\right)^2 = 1 + \frac{36i^2}{n^2}$

$\text{Right}(n) = \sum_{i=1}^{n}\left(1 + \frac{36i^2}{n^2}\right)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n} 1 + \frac{108}{n^3}\sum i^2$

$= \frac{3}{n}(n) + \frac{108}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= 3 + \frac{18(2n^2+3n+1)}{n^2}$

$= 3 + 36 + \frac{54}{n} + \frac{18}{n^2}$

$\lim_{n\to\infty} \text{Right}(n) = 3 + 36 + 0 + 0 = 39$

Answer

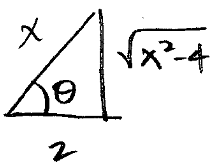
3. Do the following indefinite integrals. Show all work in complete detail.

a) (7 points) $\displaystyle\int \frac{4}{\sqrt{4-x^2}}\,dx = 4\cdot \arcsin\left(\frac{x}{2}\right) + C$

a^2 $\quad$ a

Answer

b) (7 points) $\displaystyle\int \frac{4}{x^2\sqrt{x^2-4}}\,dx = \int \frac{8\sec\theta\tan\theta\,d\theta}{4\sec^2\theta\; 2\tan\theta} = \int \frac{1}{\sec\theta}\,d\theta$



$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-4} = 2\tan\theta$

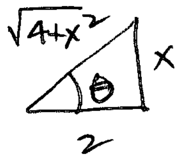
$= \int \cos\theta\,d\theta$

$= \sin\theta + C$

$= \dfrac{\sqrt{x^2-4}}{x} + C$

Answer

c) (7 points) $\displaystyle\int \frac{4}{\sqrt{4+x^2}}\,dx = \int \frac{8\sec^2\theta\,d\theta}{2\sec\theta}$



$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\sqrt{4+x^2} = 2\sec\theta$

$= \int 4\sec\theta\,d\theta$

$= 4\ln|\sec\theta + \tan\theta| + C$

$4\ln\left|\dfrac{\sqrt{4+x^2}}{2} + \dfrac{x}{2}\right| + C$

Answer

d) (8 points) $\displaystyle\int (x^2+x+1)e^{-x}\,dx = -(x^2+x+1)e^{-x} + \int (2x+1)e^{-x}\,dx$

$u = (x^2+x+1)$ $\quad dv = e^{-x}\,dx$

$du = (2x+1)\,dx$ $\quad v = -e^{-x}$

$u = 2x+1$ $\quad dv = e^{-x}\,dx$

$du = 2\,dx$ $\quad v = -e^{-x}$

$= -(x^2+x+1) - (2x+1)e^{-x} + \int 2e^{-x}\,dx$

$= -(x^2+x+1)e^{-x} - (2x+1)e^{-x} - 2e^{-x} + C$

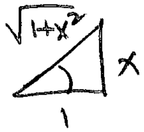
$= -(x^2+3x+4)e^{-x} + C$

Answer

4. a) (8 points) Calculate the arc length of $f(x) = \frac{1}{2}x^2 + 1$ on $[0, 1]$.

$$f'(x) = x$$

$$AL = \int_0^1 \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+x^2}\,dx = \int \sec^3\theta\,d\theta$$



$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\sqrt{1+x^2} = \sec\theta$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

$$= \frac{1}{2}\sqrt{1+x^2}\,x + \frac{1}{2}\ln\left|\sqrt{1+x^2}+x\right| \Big|_0^1$$

$$= \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln|\sqrt{2}+1| - 0$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln|\sqrt{2}+1|\right)$$

Answer

b) (8 points) Carefully determine $\int_3^5 \frac{4}{x^2-4x+3}\,dx$.

Improper

$$= \lim_{a\to 3^+} \int_a^5 \frac{2}{x-3} - \frac{2}{x-1}\,dx$$

$$\frac{4}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{Ax-3A+Bx-B}{(x-1)(x-3)}$$

$x:\ 0 = A + B$

const $: 4 = -3A - B$

$4 = -2A$

$A = -2$

$B = 2$

$$= \lim_{a\to 3^+} 2\ln|x-3| - 2\ln|x-1| \Big|_a^5$$

$$= \lim_{a\to 3^+} (2\ln 2 - 2\ln 4) - (\underbrace{2\ln|a-3|}_{-\infty} - 2\ln|a-1|)$$

Diverges

Diverges

Answer

c) (6 points) Carefully evaluate $\int_0^\infty \frac{4}{4+x^2}\,dx$.

$$= \lim_{b\to\infty} \int_0^b \frac{4}{4+x^2}\,dx = \lim_{b\to\infty} 4\cdot\frac{1}{2}\arctan\frac{x}{2}\Big|_0^b$$

$$= \lim_{b\to\infty} 2\arctan\frac{b}{2} - 2\arctan 0$$

$$= 2\left(\pi/2\right) - 0$$

π

Answer

5. (15 points) Determine each of the following limits.

a) $\lim_{x\to 0} \frac{2x - \sin(2x)}{x^2}$ $\overset{LH}{=} \lim_{x\to 0} \frac{2 - 2\cos(2x)}{2x} \overset{LH}{=} \lim_{x\to} \frac{4\sin 2x}{2} = 0$

Answer: $\boxed{0}$

b) $\lim_{x\to 0+} \left[1 + x^2\right]^{1/x}$ $= y = 1$

$\ln y = \ln\left(\lim_{x\to 0^+} [1+x^2]^{1/x}\right) = \lim_{x\to 0^+} \frac{\ln[1+x^2]}{x} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x^2}(2x)}{1} = 0$

$\ln y = 0 \Rightarrow y = e^0 = 1$

Answer: $\boxed{1}$

c) $\lim_{n\to\infty} \sqrt[n]{\left(1 + \frac{5}{n}\right)^{4n^2}}$ $= \lim_{n\to\infty} (1 + 5/n)^{4n^2/n}$

$= \lim_{n\to\infty} (1 + 5/n)^{4n} =$

$= \lim_{n\to\infty} \left[(1 + 5/n)^n\right]^4 = (e^5)^4$

Answer: $\boxed{e^{20}}$

d) $\lim_{n\to\infty} 6 + 2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} + \cdots$ $= \frac{a}{1-r} = \frac{6}{1-(-1/3)} = \frac{6}{4/3} = \frac{9}{2}$

a $r = \frac{-1}{3}$ geometric series

Answer: $\boxed{9/2}$

e) $\lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ $= \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p=1$)

Answer: $\boxed{\text{DIV}}$

6. **a)** (10 points) Carefully determine the **interval** of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^{2n}}{4^n(2n-1)}$. Justify your answer with an argument.

Ratio Ext $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-4)^{2n+2}}{4^{n+1}(2n+1)} \cdot \frac{4^n(2n-1)}{(-1)^n (x-4)^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(x-4)^2(2n-1)}{4(2n+1)} \right|$

$= \left| \frac{(x-4)^2}{4} \right| < 1 \Rightarrow |x-4|^2 < 4 \Rightarrow |x-4| < 2 = R$

Endpts: $R = 4 \pm 2 = 6, -2$

At $x=6$: $\sum_{n=1}^{\infty} \frac{(-1)^n (6-4)^{2n}}{4^n(2n-1)} = \sum \frac{(-1)^n (2)^{2n}}{4^n(2n-1)} = \sum \frac{(-1)^n 4^n}{4^n(2n-1)}$

$= \sum (-1)^n \frac{1}{2n-1}$ ... Alt series ... $\lim_{n\to\infty} \frac{1}{2n-1} = 0$

and $\frac{1}{2(n+1)-1} < \frac{1}{2n-1}$ ∴ Decr

∴ Converges by Alt series test at $x=6$

At $x=2$ $\sum \frac{(-1)^n(2-4)^{2n}}{4^n(2n-1)} = \sum \frac{(-1)^n(-2)^{2n}}{4^n(2n-1)} = \sum \frac{(-1)^n(4)^n}{4^n(2n-1)}$

$= \sum \frac{(-1)^n}{2n-1}$ ... same as above ... ~~converges~~ Converges

$(2, 6)$

Answer

b) (5 points) Carefully determine the **radius** of convergence for $\sum_{k=1}^{\infty} \frac{2^k k! x^k}{k^k}$. Justify your answer with an argument.

Ratio Test Ext

$\lim_{k\to\infty} \left| \frac{2^{k+1}(k+1)!\,x^{k+1}}{(k+1)^k} \cdot \frac{k^k}{2^k k!\, x^k} \right| = \lim_{k\to\infty} \left| \frac{2\cancel{(k+1)}\,x \quad k^k}{(k+1)^{k\cancel{+1}}} \right|$

$= \lim_{k\to\infty} \left| 2x \cdot \left(\frac{k}{k+1}\right)^k \right| = \lim_{k\to\infty} \left| 2x \left(\frac{1}{1+1/k}\right)^k \right|$

$= \left| \frac{2x}{e} \right| < 1 \Rightarrow |x| < \frac{e}{2} = R$

$e/2$

Answer

7. (10 points) Find the **AREA** enclosed by the curves $f(x) = x^3 - x^2 - 3x + 1$ and $g(x) = 5x^2 + 5x + 1$. (A graph is NOT required.)

~~Intersect~~ : $x^3 - x^2 + 3x + 1 = 5x^2 - 5x + 1$

$x^3 - 6x^2 + 8x = x(x^2 - 6x + 8) = x(x-2)(x-4) = 0$

$x = 0, 2, 4$

Top

$f(1) = 1 - 1 + 3 + 1 = 4 \quad g(1) = 5 - 5 + 1 = 1$

Top

$f(3) = 27 - 9 - 9 + 1 \quad g(3) = 45 - 15 + 1$

$$\int_0^2 \underbrace{x^3 - 6x^2 + 8x}_{f-g}\, dx + \int_2^4 \underbrace{-x^3 + 6x^2 - 8x}_{g-f}\, dx$$

$$= \frac{x^4}{4} - 2x^3 + 4x^2 \Big|_0^2 + \left(-\frac{x^4}{4} + 2x^3 - 4x^2\right)\Big|_2^4$$

$$= (4 - \cancel{16} + \cancel{16}) - 0 + (-64 + 128 - 64) - (-4 + \cancel{16} - \cancel{16})$$

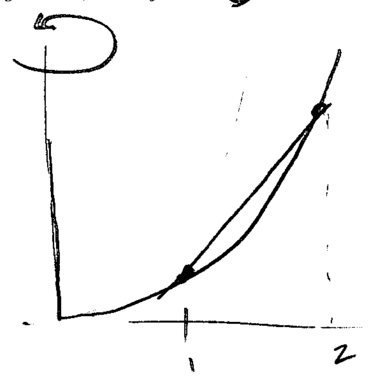
$$= 8$$

$\boxed{8}$

Answer

8. **a)** (10 points) Let R be the region **in the first quadrant** enclosed by the the y-axis, $y = x^2$, and $y = 3x + 2$. Rotate R about the y-axis. Determine the resulting **volume**. (Any method.)

$-$



$x^2 = 3x - 2 \Rightarrow x^2 - 3x + 2 = (x-2)(x-1)$

$x = 1, 2$

Shells $V = \int_1^2 2\pi x(\underbrace{3x+2}_{\text{Top}} - \underbrace{x^2}_{\text{Bot}})\, dx$

$= \int_1^2 2\pi \left[3x^2 + 2x - x^3\right] dx$

$= 2\pi \left[x^3 + x^2 - \frac{x^4}{4}\right]\Big|_1^2$

$= 2\pi\left[(8 + \cancel{4} - \cancel{4}) - (1 + 1 - 1/4)\right] = 2\pi(9\tfrac{3}{4}) = 19\tfrac{1}{2}\pi$

$\boxed{19\tfrac{1}{2}\pi}$

Answer

b) (5 pts) Let R be the same region as in part (a). Rotate R about the x-axis. **Set up** the integral for the resulting **volume**. (DO NOT EVALUATE THE INTEGRAL.)

Disks

outside   inside

$$V = \int_1^2 \pi(3x+2)^2\, dx - \int_1^2 \pi(x^2)^2\, dx$$

Answer

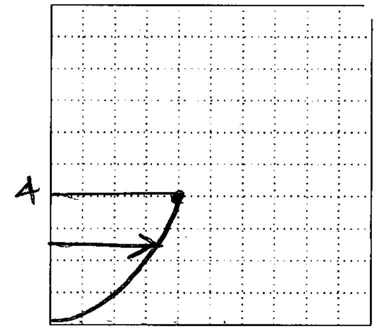
9. (10 points) Let R be the region in the first quadrant enclosed by $y = \frac{1}{4}x^2$, the y-axis, and $y = 4$. Revolve R around the y-axis to form a shallow tank. The tank has oil in it with density of 60 lbs/ft^3. If the depth of the oil is 3 feet, calculate the work done in pumping all of the oil to a height 2 feet above the top edge of the tank.

Cross-sect Area $= \pi r^2 = \pi x^2$

$= \pi 4y$

2ft above top

$$W = 60 \int_0^3 \pi x^2 (6-x)\, dx$$



$$= 60\pi \int_0^3 6x^2 - x^3\, dx$$

$$= 60\pi \left[2x^3 - \frac{x^4}{4}\right]_0^4 = 60\pi\,[128 - 64]$$

$$= 60 \cdot (64)\pi$$

$$= 3840\pi$$

10. (9 points) Determine whether the following **arguments** are correct. Answer 'correct' if the argument is completely correct. Answer 'incorrect' if there is a mistake in the argument (indicate where the error is) even in the final answer is correct.

a) Using u-substitution, $\displaystyle\int_{-5}^{5} \frac{x}{\sqrt{x^2-9}}\, dx = \sqrt{x^2-9}\Big|_{-5}^{5} = \sqrt{16} - \sqrt{16} = 0.$

No... improper @ $x = \pm 3$

b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+2}$ diverges by direct comparison since $0 < \frac{1}{5n+2} < \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test.

No To use direct comparison here we would want the terms of the unknown series to be larger than 1/n to get divergence

c) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n^2)}$ converges by the nth term test since $\displaystyle\lim_{n\to\infty} \frac{1}{\arctan(n^2)} = 0.$

$= \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi} \neq 0$

No

Even if $\lim\limits_{n\to\infty} a_n = 0$, this test does not give convergence

$(-1)^n$

11. a) (10 points) Carefully determine whether $\sum_{n=1}^{\infty} \frac{\cos(n\pi)(4n^3+1)}{n^4}$ converges absolutely, conditionally, or diverges. Justify your answer with an argument.

Check Abs Conv $\sum \left| \frac{\cos(\pi n)(4n^3+1)}{n^4} \right| = \sum \frac{4n^3+1}{n^4}$. Compare to $\sum \frac{1}{n}$

Both have positive terms

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^3+1}{n^4} \cdot \frac{n}{1} \stackrel{HP}{=} 4$. Since $\sum \frac{1}{n}$ diverges

so does $\sum \left| \frac{\cos(\pi n)(4n^3+1)}{n^4} \right|$ by limit Comp.

Not absolutely convergent,

use Alt. Series Test

$a_n = \frac{4n^3+1}{n^4} > 0$. ① $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^3+1}{n^4} \stackrel{HP}{=} 0$ ✓

② Decr? $f(x) = \frac{4x^3+1}{x^4}$; $f'(x) = \frac{12x^2 x^4 - 4x^3(4x^3+1)}{x^8}$

$= -4x^6 - 4x^3 < 0$ when $x \geq 1$ ∴ Decreasing

∴ Converges

By Alt series Test, the series is conditionally convergent

Conditional

Answer

b) (5 points) Carefully determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n[(n+1)!]^3}{(3n)!}$ converges. Justify your answer with an argument.

Try Ratio Test Ext:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}[(n+2)!]^3}{(3n+3)!} \cdot \frac{(3n)!}{(-1)^n[(n+1)!]^3} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)}$$

$$\stackrel{HP}{=} \lim_{n\to\infty} \frac{n^3}{27n^3} = \frac{1}{27} < 1$$

By the ratio Test Ext, the series converges (absolutely)

Answer

12. (8 points) Let f be the function whose graph is given below. Use the information in the table, properties of the integral, and the **shape** of f to evaluate the given integrals.

a) $\int_3^0 f(x)\,dx$ **b)** $\int_1^4 5+2f(x)\,dx$ **c)** $\int_{-4}^4 f(x)+3\,dx$ **d)** $\int_{-1}^2 f(x)\,dx$

a) $\int_3^0 f(x)dx = -\int_0^3 f(x)dx = -0.9$

b) $\int_1^4 5+2f(x)dx = \int_1^4 5dx + 2\int_1^4 f(x)dx$

$= 5x\Big|_1^4 + 2\left[\int_0^4 f(x)dx - \int_0^1 f(x)dx\right]$

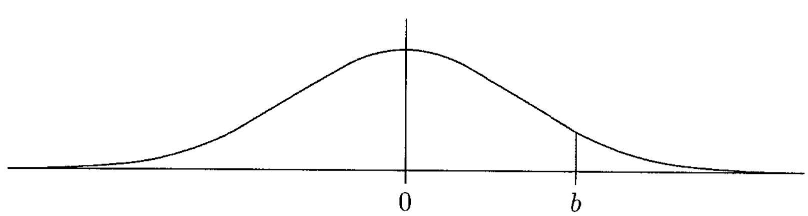
$= 5(4-1) + 2[1-.4] = 16.2$

c) $\int_{-4}^4 f(x)+3\,dx \underset{\text{sym}}{=} 2\int_0^4 f(x)dx + \int_{-4}^4 3dx$

$= 2[1] + 3x\Big|_{-4}^4 = 2 + 3(4-(-4)) = 26$

symmetry

d) $\int_{-1}^2 f(x)dx = \int_0^1 f(x)dx + \int_0^4 f(x)dx = .4 + 1 = 1.4$



$\int_0^1 f(x)\,dx = 0.4$
$\int_0^2 f(x)\,dx = 0.8$
$\int_0^3 f(x)\,dx = 0.9$
$\int_0^4 f(x)\,dx = 1.0$

14. **a)** (6 points) Determine $\int \cos^6(4x)\sin^3(4x)\,dx = \int \cos^6(4x)\sin^2(6x)\sin(6x)\,dx$

$$= \int \cos^6(4x)(1-\cos^2 4x)\sin(4x)\,dx$$

$u = \cos 4x$

$du = -4\sin 4x\,dx$

$-\frac{1}{4}du = \sin 4x\,dx$

$$= -\frac{1}{4}\int u^6(1-u^2)\,du$$

$$= -\frac{1}{4}\int u^6 - u^8\,du$$

$$= -\frac{1}{4}\left[\frac{1}{7}u^7 - \frac{1}{9}u^9\right] + C$$

$$= -\frac{1}{28}\cos^7(4x) + \frac{1}{36}\cos^9(4x) + C$$

b) (8 points) Determine $\int \frac{-4x+4}{(x-2)^2 x}\,dx$

$$\frac{-4x+4}{(x-2)^2 x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{Ax^2 - 2Ax + 4A + Bx^2 - 2Bx + Cx}{x(x-2)^2}$$

x^2: $0 = A + B$

x: $-4 = -2A - 2B + C$

Const: $4 = 4A$

$A = 1,\ B = -1,\ C = -4$

$$\int \frac{1}{x} - \frac{1}{x-2} - \frac{4}{(x-2)^2}\,dx = \ln|x| - \ln|x-1| + 4(x-2)^{-1} + C$$