Math 131 Lab 12: Series

1. **Warmup:** Determine whether the series \( \sum_{n=1}^{\infty} \frac{n+1}{n!} \) converges. Give an argument.

2. Determine whether the following series converge. First determine which test to use for each one: Divergence (nth) term, geometric, p-series, ratio, or integral test. Your final answer should consist of a little ‘argument’ (a sentence or two) and any necessary calculations. Use appropriate mathematical language. Here’s a Model Example:

   Does \( \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)} \) converge?

   **Preliminary Analysis—Scrap Work:** Think about it. Try the easy tests first: Notice that this is not a geometric series or p-series and the Divergence (nth) term test fails (\( a_n \to 0 \)). The ratio test seems inappropriate, no factorials or powers. So we are left with the integral test. Now here’s what you might write:

   **ARGUMENT:** Use the integral test. The corresponding function is \( f(x) = \frac{1}{(x+1)\ln(x+1)} \) which is positive, decreasing (as \( x \) gets bigger, so does the denominator but the numerator stays the same, so the fraction gets smaller), and it is continuous on \([1, \infty)\). The improper integral that we must evaluate is \( \int_{1}^{\infty} \frac{1}{(x+1)\ln(x+1)} \, dx \).

   Using a \( u \)-substitution with \( u = \ln(x+1) \) and \( du = \frac{1}{x+1} \, dx \) check that

   \[
   \int_{1}^{\infty} \frac{1}{(x+1)\ln(x+1)} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{(x+1)\ln(x+1)} \, dx = \lim_{b \to \infty} \ln|\ln(x+1)|_{1}^{b} = \lim_{b \to \infty} \ln|\ln(b+1)| - \ln(\ln(2)) = +\infty.
   \]

   Since the integral diverges the integral test says the series \( \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)} \) also diverges.

   a) \( \sum_{n=1}^{\infty} \frac{1}{n^{1001}} \)  
   b) \( \sum_{n=1}^{\infty} \frac{5 \cdot n!}{2^n} \)  
   c) \( \sum_{n=1}^{\infty} \frac{2}{1+4n^2} \)  
   d) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}^n} \)  

   e) \( \sum_{n=1}^{\infty} \ln(2n+3) - \ln(3n+2) \)  
   f) \( \sum_{n=1}^{\infty} \frac{5^n}{(n+1)!} \)  
   g) \( \sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{5} \)  
   h) \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \)  

   i) \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)  
   j) \( \sum_{n=1}^{\infty} \frac{5^n}{3^n} \)  
   k) \( \sum_{n=1}^{\infty} 2 \arctan(n) \)  
   l) \( \sum_{n=1}^{\infty} (-1)^n \)  

   m) \( \sum_{n=1}^{\infty} \frac{6}{n} \)  
   n) \( \sum_{n=1}^{\infty} \frac{5^n}{2^n} \)  
   o) \( \sum_{n=1}^{\infty} \frac{n^4 - 1}{n^4} \)  
   p) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}^6} \)  

   q) \( \sum_{n=0}^{\infty} \frac{2n}{n^2 + 1} \)  
   r) \( \sum_{n=0}^{\infty} \frac{3n}{n^2 + 1} \)  
   s) \( \sum_{n=1}^{\infty} \frac{10}{n^2 + 5n} \)  
   t) \( \sum_{n=1}^{\infty} 3 \cdot (-2)^n \cdot 7^{-n} \)  

3. a) The Divergence (nth) term test says that if \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges. Does this mean that if \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) converges? Explain your answer. (See the next parts)

   b) Give two examples of a series \( \sum a_n \) where \( \lim a = 0 \) and the series diverges.

   c) Give two examples of a series \( \sum a_n \) where \( \lim a = 0 \) and the series converges.

4. Determine whether the following geometric series converge. If so, to what? (Watch the starting indices.)

   a) \( \sum_{n=2}^{\infty} -4 \left( \frac{2}{5} \right)^n \)  
   b) \( \sum_{n=0}^{\infty} 2 \left( -\frac{5}{3} \right)^n \)  
   c) \( \sum_{n=0}^{\infty} \frac{2^n}{3^n+3} \)  
   d) \( \sum_{n=1}^{\infty} 3 \cdot (-2)^n \cdot 7^{-n} \)

5. **Extra Credit:** Determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \) converges.
**Brief Answers**

Full answers will be available online.


2. The simplest test to apply... Your answers will include calculations and explanations as in the **ARGUMENT** on the other side of the page. Ask me to check your work.
   
   a) \( p \)-series
   b) Ratio Test
   c) Integral test
   d) \( p \)-series test
   e) Divergence \((n\text{th})\) term test
   f) Ratio Test
   g) Divergence \((n\text{th})\) term test
   h) Geometric Series Test
   i) Integral test
   j) Ratio Test
   k) Divergence \((n\text{th})\) term test
   l) Geometric Series Test
   m) Ratio Test
   n) Geometric Series Test
   o) Divergence \((n\text{th})\) term test
   p) \( p \)-series test
   q) Integral test
   r) Divergence \((n\text{th})\) term test
   s) Integral test
   t) Geometric Series Test

3. a) No. When \( \lim_{n \to \infty} a_n = 0 \) the series may converge as it does with the \( p \)-series \( \sum \frac{1}{n^2} \) where \( 2 = p > 1 \). But it could diverge when \( \lim_{n \to \infty} a_n = 0 \) the series may diverge as it does with harmonic series \( \sum \frac{1}{n} \) where \( 1 = p \leq 1 \).

4. Remember a geometric series has the form \( \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots \). Write out the first few terms to determine \( a \) and \( r \).
   
   a) \( -\frac{16}{15} \)
   b) Diverges.
   c) \( \frac{5}{7} \)
   d) \( -\frac{2}{3} \).
1. **ARGUMENT:** Factorial: Ratio test. The terms are positive. \[ r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n + 2}{n + 2} = \frac{1 + \frac{2}{n^2}}{1 + \frac{2}{n^2}} = 0. \text{ Since } r < 1 \text{ by the ratio test the series converges.} \]

   \[ \lim_{n \to \infty} \frac{n + 2}{n^2 + 2n + 1} = \lim_{n \to \infty} \frac{1/n^2 + 2/n + 1/n^2}{1 + 2/n + 1/n^2} = 0. \]

2. a) **ARGUMENT:** Converges by the p-series test; \( p = 1.0101 > 1. \)
   
   b) HW
   
   c) HW
   
   d) **ARGUMENT:** Diverges by the p-series test; \( p = \frac{3}{2} \leq 1. \)
   
   e) **ARGUMENT:** The Divergence (nth) term test.
   
   \[ \lim_{n \to \infty} \ln(2n + 3) - \ln(3n + 2) = \lim_{n \to \infty} \ln \left( \frac{2n + 3}{3n + 2} \right) = \lim_{n \to \infty} \ln \left( \frac{2 + \frac{3}{n}}{3 + \frac{2}{n}} \right) = \ln \frac{2}{3} \neq 0. \]
   
   By the Divergence (nth) term test the series diverges.
   
   f) **ARGUMENT:** Factorial: Ratio test. The terms are positive. \[ r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{5^{n+1}}{n+2}}{\frac{5^n}{n}} = \frac{5}{n+2} \to 0. \text{ Since } r < 1 \text{ by the ratio test the series converges.} \]
   
   g) **ARGUMENT:** Divergence (nth) term test: \( \lim_{n \to \infty} \frac{1}{n} = \sec(0) = 1 \neq 0. \) By the Divergence (nth) term test the series diverges.
   
   h) **ARGUMENT:** Geometric Series Test: This is a geometric series with \( |r| = \frac{2}{3} < 1. \) By the geometric series it converges.
   
   i) **ARGUMENT:** Integral test: Note that \( \frac{1}{\ln x} \) is positive and continuous on \( [2, \infty) \). It is also decreasing because as \( x \) increases, the denominator increases, but the numerator stays the same making the function values smaller. Let \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \).
   
   \[ \int_2^\infty \frac{1}{x \ln x} \, dx = \int_\infty^0 \frac{1}{u} \, du = \ln |u| = \ln |\ln x|. \]
   
   So
   
   \[ \int_2^\infty \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \ln |\ln u| \bigg|_2^b = \lim_{b \to \infty} \ln |\ln b| - \ln |\ln 2| = \infty. \]
   
   Since \( \int_2^\infty \frac{1}{x \ln x} \, dx \) diverges so does \( \sum_{n=2}^\infty \frac{1}{n \ln n} \) by the integral test.
   
   j) **ARGUMENT:** Factorial: Ratio test. The terms are positive. \[ r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{(n+1)^{n+1}}{n^{n+1}}}{\frac{3}{n!}} = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^n = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e > 1. \text{ Since } r > 1 \text{ by the ratio test the series diverges.} \]
   
   k) HW
   
   l) **ARGUMENT:** Geometric Series Test: Here \( |r| = | -1 | = 1. \) Diverges by the geometric series test. Or use the Divergence (nth) term test: \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n DNE \neq 0. \) So by the Divergence test the series diverges.
   
   m) **ARGUMENT:** Powers: Ratio test. The terms are positive. \[ r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)e^{-n-1}}{ne^{-n}} = \lim_{n \to \infty} \frac{(n+1)e^{-1}}{n} = e^{-1} < 1. \text{ Since } r < 1 \text{ by the ratio test the series converges. (This is actually easier to do by the root test, which we will cover next.)} \]
   
   n) **ARGUMENT:** Geometric Series Test: This is a geometric series with \( |r| = \frac{5}{2} > 1. \) So it diverges. (Or use the nth term test.)
   
   o) **ARGUMENT:** Divergence (nth) term test. \[ \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} 1 - \frac{1}{n} = 1 \neq 0. \] By the Divergence (nth) term test the series diverges.
   
   p) HW
q) **ARGUMENT:** Integral test: Note that \( \frac{2x}{x^2+1} \) positive, continuous, and decreasing since \( f'(x) = \frac{1-2x^2}{(x^2+1)^2} < 0 \) on \([1, \infty)\). Let \( u = x^2 + 1 \). Then \( du = 2x \, dx \). So

\[
\int_1^\infty \frac{2x}{x^2+1} \, dx = \lim_{b \to \infty} \int_1^b \frac{2x}{x^2+1} \, du = \lim_{b \to \infty} \ln |x^2 + 1| \bigg|_1^b = \lim_{b \to \infty} \ln |b^2 + 1| - \ln 1 = \infty.
\]

Since \( \int_1^\infty \frac{2x}{x^2+1} \, dx \) diverges, so does \( \sum_{n=1}^{\infty} \frac{2n}{n^2+1} \) by the integral test.

r) **ARGUMENT:** Divergence (nth) term test. Remember if \( f(x) = a^n \), then \( f'(x) = (\ln a) x^n \). So

\[
\lim_{n \to \infty} \frac{3^n}{n^2+1} = \lim_{x \to \infty} \frac{3^x}{x^2+1} = \lim_{x \to \infty} \frac{(\ln 3)^x}{2x} = \lim_{x \to \infty} \frac{(\ln 3)^x}{2} = \infty \neq 0.
\]

By the Divergence (nth) term test the series diverges.

s) **ARGUMENT:** The integral test. Note that \( f(x) = \frac{10}{x^2+5x} \) is positive and continuous on \([1, \infty)\). It is also decreasing because as \( x \) increases, the denominator increases, but the numerator stays the same making the function values smaller. Or use \( f'(x) = -10(2x+5)(x^2+5x)^{-2} < 0 \) on \([1, -\infty)\). Use partial fractions.

\[
\int_1^\infty \frac{10}{x^2+5x} \, dx = \lim_{b \to \infty} \int_1^b \frac{2}{x} - \frac{2}{x+5} \, dx = \lim_{b \to \infty} 2 \ln |x| - 2 \ln |x+5| \bigg|_1^b = \lim_{b \to \infty} 2 \ln \frac{x}{x+5} \bigg|_1^b
\]

\[
= \lim_{b \to \infty} 2 \ln \frac{b}{b+5} - 2 \ln \frac{1}{6} = \lim_{b \to \infty} 2 \ln \frac{1}{1+\frac{5}{b}} - 2 \ln \frac{1}{6} = 2 \ln 1 - 2 \ln \frac{1}{6} = \ln 36.
\]

Since the integral converges, so does the corresponding series \( \sum_{n=1}^{\infty} \frac{10}{n^2+5n} \) by the integral test.

t) HW

3. No. For example the series \( \sum \frac{1}{n} \) diverges by the p-series test. But \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0 \). So even though the Divergence (nth) term \( \to 0 \), the series still diverges.

4. Remember a geometric series has the form \( \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots \). Write out the first few terms to determine \( a \) and \( r \).

a) \( \sum_{n=2}^{\infty} -4 \left( \frac{2}{5} \right)^n = -\frac{16}{25} - \frac{32}{125} - \frac{64}{625} \cdots \), \( a = -\frac{16}{25}, r = \frac{2}{5} \). Sum: \( \frac{-16}{1-\frac{2}{5}} = \frac{-16}{15} \).

b) Diverges since \( |r| = \frac{2}{5} > 1 \).

c) \( \sum_{n=0}^{\infty} \frac{2^n}{3^n+3} = \frac{\frac{2}{7}}{a} + \frac{\frac{10}{81}}{ar} + \frac{\frac{20}{243}}{ar^2} + \cdots \), \( a = \frac{5}{27}, r = \frac{a_{n+1}}{a_n} = \frac{\frac{10}{81}}{\frac{5}{27}} = \frac{2}{3} \). Sum: \( \frac{\frac{5}{27}}{1-\frac{2}{3}} = \frac{5}{9} \).

d) \( \sum_{n=1}^{\infty} 3 \cdot (-2)^n \cdot 7^{-n} = -\frac{6}{7} + \frac{12}{49} - \frac{24}{343} + \cdots \), \( a = -\frac{6}{7}, r = \frac{a_{n+1}}{a_n} = \frac{\frac{12}{49}}{-\frac{6}{7}} = -\frac{2}{7} \). Sum: \( \frac{-\frac{6}{7}}{1-\frac{2}{7}} = -\frac{2}{3} \).

5. Use the integral test with triangles. \( x = \tan \theta, \, dx = \sec^2 \theta \, d\theta \), and \( \sqrt{x^2+1} = \sec \theta \). So

\[
\int \frac{1}{\sqrt{x+1}} \, dx = \int \frac{\sec^2 \theta}{\sec \theta} \, d\theta = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + c = \ln |\sqrt{x^2+1} + x| + c.
\]

So

\[
\int_1^\infty \frac{1}{\sqrt{x+1}} \, dx = \lim_{b \to \infty} \ln |\sqrt{b^2+1} + b| - \ln |\sqrt{2} + 1| = \infty.
\]

Since the integral diverges, so does the corresponding series by the integral test.