Techniques of Integration

In this chapter we will expand our toolkit of integration techniques. At this point the only technique, other than just recognizing an antiderivative, that we have developed is u-substitution. By the time we are finished it will require some insight to choose the correct technique for each problem. Sometimes there will be more than one method that works, though one technique may be much simpler than another.

Remember that integration reverses differentiation. In particular u-substitution reverses the chain rule for derivatives. We begin this chapter by exploiting a technique that reverses another of the basic derivative rules, the product rule.

8.1 Integration by Parts: Reversing the Product Rule

Because integration reverses differentiation, every derivative rule can be 'reversed' to become an integration rule. Sometimes this is easy; other times it is not. In this section we will reverse the product rule for derivatives. Let's set up the situation.

Suppose that $u(x)$ and $v(x)$ are both differentiable functions. Then the product rule says

$$\frac{d}{dx}\left[u(x)v(x)\right] = u(x)v'(x) + u'(x)v(x).$$

A short-hand way to write this is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{8.1}$$

Of course, if we integrate both sides of (8.1), we get

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx.$$

On the left side of this equation integration undoes differentiation, and on the right side we can simplify the notation as we did in substitution problems so that we end up with

$$uv = \int u\,dv + \int v\,du.$$

Solving for the first term on the right side, we find

$$\int u\,dv = uv - \int v\,du.$$

We have used the product rule to develop a new antidifferentiation rule. This result is sufficiently important that we single it out as a theorem.

THEOREM 8.1.1 (Integration by Parts Formula). Suppose that $u(x)$ and $v(x)$ are both differentiable functions. Then

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Take-Home Message. Here's how to read Theorem 8.1.1: We can trade one integral that we cannot do, say $\int u\,dv$ for another that we can (might be able) to do, namely, $\int v\,du$. To do this we need to be able to identify a function u and the derivative of another function v in the original integrand. This technique is especially useful when an integrand contains two functions that are not related to each other. By the way, it is called 'integration by parts' because u and v are the 'parts.' Let's take a look at some examples.

EXAMPLE 8.1.2. Consider the integral

$$\int x\cos(x)\,dx.$$

The function x is unrelated to the function $\cos(x)$. Compare this to

$$\int x\cos(x^2)\,dx.$$

In this second situation, the function x is related to $\cos(x^2)$. In particular, x is almost the derivative of the 'inside function' x^2—we are off only by a constant. You should recognize this second problem as a u-substitution problem. But what about the first integral?

Solution. Let's see how to use integration by parts to solve the integral. In the original integral $\int x\cos(x)\,dx$ we have to identify a function u and the derivative of another function v. Let's try

$$u = x \quad \text{and} \quad dv = \cos x\,dx$$

The integration by parts formula then says $\int u\,dv = uv - \int v\,du$. So we need to determine v and du from the information we have so far. Getting du is easy; just take the derivative of u:

$$u = x \Rightarrow du = 1dx = dx.$$

Getting v requires us to integrate dv:

$$v = \int dv = \int \cos x\,dx = \sin x.$$

Notice that we have not added $+c$ to the solution for v. As we'll see below, it turns out not to matter in this situation. Thus $uv = x\sin(x)$ and $\int v\,du = \int \sin(x)\,dx$. Notice that we are able to do this latter integral! We have exchanged the integral we could not determine, $\int x\cos(x)\,dx$, for one that we can, $\int \sin(x)\,dx$.

Substituting all of this into the integration by parts formula produces

$$\int u\,dv = uv - \int v\,du \Rightarrow \int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + c.$$

We can check that this is correct by differentiating our answer:

$$\frac{d}{dx}\left(x\sin(x) + \cos(x) + c\right) = \sin(x) + x\cos(x) - \sin(x) = x\cos(x),$$

which is the original integrand. Notice how the product rule comes into play in checking the answer.

'What if' #1. What if we had chosen 'the parts' differently. Suppose we let $u = \cos x$ and $dv = x\,dx$. Then

$$u = \cos x \Rightarrow du = -\sin x\,dx.$$

and

$$v = \int dv = \int x\,dx = \frac{x^2}{2}.$$

Substituting all of this into the integration by parts formula produces

$$\int u\,dv = uv - \int v\,du \Rightarrow \int x\cos(x)\,dx = \frac{x^2}{2}\cos(x) - \int -\frac{x^2}{2}\sin(x)\,dx$$

which we still do not know how to integrate. In fact, if anything, the new integral is 'worse' than the original one because of the x^2-term. The power of x has increased. In our original solution, the power of x decreased because we let u be the function x. Its derivative $du = dx$ appeared in the new integrand (as well as v), so the new integral was simpler. We will list a few guidelines below that may help in selecting the 'parts.'

'What if' #2. What if we had added a constant $+C$ to v in the original solution (so that $v + C = \sin x + C$). Let's see what happens in the general integration by parts formula when we substitute $v + c$ for v.

$$\begin{aligned}\int u\,dv = u(v+C) - \int (v+C)\,du &= uv + Cu - \int v\,du - \int C\,du \\ &= uv + Cu - \int v\,du - Cu \\ &= uv - \int v\,du.\end{aligned}$$

We get the same answer as in the original integration by parts formula because Cu and $-Cu$ end up canceling each other. In most cases it is simpler to just ignore the $+C$.

EXAMPLE 8.1.3 (Classic Parts)**.** Determine $\int xe^{-x}\,dx$.

Solution. We have unrelated functions x and e^{-x} so integration by parts may be useful, since our other technique of substitution does not make sense here. We present the solution in tabular form which shows the parts in the left column and the integration in the right column.

$u = x$	$dv = e^{-x}\,dx$	$\int u\,dv = uv - \int v\,du$
$du = dx$	$v = \int dv = \int -e^{-x}\,dx = e^{-x}$	$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$
		$= -xe^{-x} - e^{-x} + c$

We can check that this is correct by differentiating our answer:

$$\frac{d}{dx}\left(-xe^{-x} - e^{-x} + c\right) = -e^{-x} + xe^{-x} + e^{-x} = xe^{-x},$$

which is the original integrand.

EXAMPLE 8.1.4. Determine $\int \ln x\,dx$.

Solution. Careful, we don't want the derivative of $\ln x$, we want the antiderivative. This time we have no choice. Since we don't know how to integrate $\ln x$

yet, we have to differentiate it. In other words we want $u = \ln x$ and that leaves $dv = dx$. So

$u = \ln x$	$dv = dx$	$\int u\,dv = uv - \int v\,du$
$du = \frac{1}{x}\,dx$	$v = \int dv = \int dx = x$	$\int \ln x\,dx = x\ln x - \int 1\,dx = x\ln x - x + c$

We can check that this is correct by differentiating our answer:

$$\frac{d}{dx}\left(x\ln x - x + c\right) = \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x,$$

which is the original integrand.

Some Tips for Using Integration by Parts

First, remember that this technique is most useful when there are unrelated functions in the integrand. Second, right now you have only two techniques: substitution and parts. Substitution is usually easier to carry out, so first check to see if that technique applies. At this point, if it does not, then parts is likely to be the method that works. Third, if you choose parts and the integral gets 'worse' after the first substitution, step back and ask yourself whether the parts should be chosen differently. Be prepared to change; you may not always choose the parts correctly the first time.

Here are a couple of general principles that will help you select u and dv when using integration by parts. There really is no substitute for practice, but these may help you be more efficient.

- Try selecting dv as the most complicated portion of the integrand that you can integrate... and let u be the rest of the integrand.
- Try letting u be the portion of the integrand whose derivative is simpler than u, e.g., du might end up being a lower power of x. This is what we did in Example 8.1.2 when we selected $u = x$. There $du = 1\,dx$ was simpler. If we had chosen $u = \cos x$, then $du = -\sin x\,dx$ would just as complicated as u. The new integrand would not have been simplified.

Typical Scenarios for Integration by Parts

The next few examples illustrate several basic situations where integration by parts is useful.

EXAMPLE 8.1.5 (Unrelated Parts). Determine $\int \sqrt{x}\ln x\,dx$.

Solution. The two functions in the integrand, $\sqrt{x}$ and $\ln x$ are not related to each other. Parts should immediately come to mind. Since $\ln x$ is not simple to integrate (though we know how to do it from Example 8.1.4), it makes sense to use it as u. So

$u = \ln x$	$dv = \sqrt{x}\,dx$	$\int u\,dv = uv - \int v\,du$
$du = \frac{1}{x}\,dx$	$v = \int dv = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2}$	$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx$
		$= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}\,dx$
		$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + c$

You should check that this is correct by differentiating the answer.

WEBWORK: Click to try Problems 99 through 101.