Day 38 Math 131

Show me the absolute values

#1 a) $\sum \frac{(-1)^n}{\sqrt[3]{n^4+1}}$ Check Absolute Convergence First

or limit comp

So $\sum \left|\frac{(-1)^n}{\sqrt[3]{n^4+1}}\right| = \sum \frac{1}{\sqrt[3]{n^4+1}} \sim \sum \frac{1}{n^{4/3}}$ ... use direct comparison

compare: $0 < \frac{1}{\sqrt[3]{n^4+1}} < \frac{1}{n^{4/3}}$. But $\sum \frac{1}{n^{4/3}}$ converges ($p = 4/3 > 1$, p-series)

So by direct comparison $\sum \frac{1}{\sqrt[3]{n^4+1}}$ also converges.

So the original series $\sum \frac{(-1)^n}{\sqrt[3]{n^4+1}}$ converges absolutely.

DONE

b) $\sum^{\infty} \frac{\cos(n)}{\sqrt{n+10}}$ Check Absolute Convergence. $\sum \left|\frac{\cos(n)}{\sqrt{n+10}}\right| = \sum \frac{1}{\sqrt{n+10}}$

Use limit comparison w/ $\sum \frac{1}{n^{1/2}}$. Both series are positive

$L = \lim_{n\to\infty} \frac{1}{\sqrt{n+10}} \cdot \frac{n^{1/2}}{1} \overset{(HP)}{=} 1$ Since $0 < L = 1 < \infty$, and since $\sum \frac{1}{n^{1/2}}$ diverges (p-series, $p = 1/2 \leq 1$), then $\sum \frac{1}{\sqrt{n+10}}$ diverges. We Must Check Conditional Convergence w/ Alt. Series Test

✓ $a_n = \frac{1}{\sqrt{n+10}} > 0$. Check 2 conditions:

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+10}} = 0$ ✓

2. Decr? $f(x) = (x+10)^{-1/2}$; $f'(x) = -\frac{1}{2}(x+10)^{-3/2} < 0$ ✓ $(x \geq 1)$

so the sequence is decreasing.

Therefore by Alt Series test, the series converges

$\boxed{\text{Conditional Convergence}}$ since it does not converge abs.

c) $\sum \frac{(-7)^{k+1}}{k!}$ Check abs Convergence: $\sum \left|\frac{(-7)^{k+1}}{k!}\right| = \sum \frac{7^{k+1}}{k!}$

The terms are positive - Factorial - Ratio Test

$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{7^{k+2}}{(k+1)!} \cdot \frac{k!}{7^{k+1}} = \lim_{k\to\infty} \frac{7}{k+1} = 0 < 1$

Since $r < 0$, the series converges (absolutely) by the ratio test. We are done!

Day 38 Math 131

#2 a) $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ Check Abs Convergence first. $\sum \left|\frac{(-1)^k}{\ln k}\right| = \sum \frac{1}{\ln k}$

Use direct (or limit) comparison w/ $\sum 1/n$. Notice $0 < \frac{1}{n} < \frac{1}{\ln n}$

Since $\sum \frac{1}{n}$ diverges, by direct comparison $\sum \frac{1}{\ln n}$ <u>diverges</u>
So NOT absolutely convergent ...
Check Conditional Convergence w/ Alt. Series Test

✓ $a_n = \frac{1}{\ln n} > 0 \quad (n \geq 2)$

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{\underbrace{\ln n}_{\to\infty}} = 0$ ✓
2. decreasing? Yes, the numerator is constant and the denominator increases as n does. ✓

By the alternating Series Test, $\sum \frac{(-1)^n}{\ln n}$ converges

The series <u>converges conditionally</u> (Does not converge abs)

#2(b) $\sum (-1)^n \frac{(n!)^3}{(3n)!}$ Check Abs Conv: $\sum \left|\frac{(-1)^n (n!)^3}{(3n)!}\right| = \sum \frac{(n!)^3}{(3n)!}$

Terms are positive – Ratio Test

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)}$$

$$\overset{HP}{=} \lim_{n\to\infty} \frac{n^3}{27n^3} = \frac{1}{27} < 1$$

. Since $r < 1$, the series converges by the ratio test. The series converges absolutely. Done

$$\text{Notice: } \frac{[(n+1)!]^3 \cdot (3n)!}{(3n+3)!\ (n!)^3} = \frac{(n+1)!\,(n+1)!\,(n+1)!\,(3n)!}{n! \cdot n! \cdot n! \cdot (3n+3)!}$$

3 more terms

$$= \frac{(n+1)(n+1)(n+1)}{(3n+1)(3n+2)(3n+3)}$$

$$= \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} \overset{HP}{=} \frac{n^3}{27n^3}$$

Day 38 Math 131

#3 $f(x) = e^{-x}$. Find $p_5(x)$ @ $a=0$. Find $p_n(x)$

$$\begin{aligned}
f(x) &= e^{-x} & f(0) &= 1\\
f'(x) &= -e^{-x} & f'(0) &= -1\\
f''(x) &= e^{-x} & f''(0) &= +1\\
f'''(x) &= -e^{-x} & f'''(0) &= -1\\
f^{(4)}(x) &= e^{-x} & f^{(4)}(0) &= +1\\
f^{(5)}(x) &= -e^{-x} & f^{(5)}(0) &= -1
\end{aligned}$$

$$\begin{aligned}
p_5(x) &= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \dots + \frac{f^{(5)}(0)(x-0)^5}{5!}\\
&= 1 - 1x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5
\end{aligned}$$

pattern $f^{(k)}(0) = (-1)^k$

$$p_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^k$$

#4 $f(x) = \ln(x+1)$. Find $p_4(x)$ @ $a=0$. Find $p_n(x)$

$$\begin{aligned}
f(x) &= \ln(x+1) & f(0) &= 0\\
f'(x) &= \frac{1}{x+1} = (x+1)^{-1} & f'(0) &= 1\\
f''(x) &= -(x+1)^{-2} = -1!(x+1) & f''(0) &= -1\\
f'''(x) &= +2(x+1)^{-3} = 2!(x+1)^2 & f'''(0) &= 2!\\
f^{(4)}(x) &= -3\cdot 2(x+1)^{-4} = -3!(x+1)^3 & f^{(4)}(0) &= -3!
\end{aligned}$$

pattern $f^{(k)}(0) = (-1)^{k-1}(k-1)!$

$$\Longrightarrow p_4(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{3!}{4!}x^4$$

$$= 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

$$p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$$