Day 40 Math 131

#1a) $f(x) = (1-x)^{-1}$. Find $p_4(x)$ with $a=0$

$$\left.\begin{array}{ll} f(x) = (1-x)^{-1} & f(0) = 1 = 0! \\ f'(x) = (1-x)^{-2} & f'(0) = 1 = 1! \\ f''(x) = 2(1-x)^{-3} & f''(0) = 2 = 2! \\ f'''(x) = 3!(1-x)^{-4} & f'''(0) = 3! \\ f^{(4)}(x) = 4!(1-x)^{-5} & f^{(4)}(0) = 4! \end{array}\right\} f^{(k)}(0) = k!$$

$$p_4(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^2 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$p_4 = 1 + x + x^2 + x^3 + x^4$$

(b) $$p_\infty = \sum_{k=0}^{\infty} \frac{k!}{k!}(x-0)^k = \sum_{k=0}^{\infty} x^k$$

Bonus: radius of convergence: Ratio Test

$$\lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| = |x| < 1 \ldots \boxed{R=1}$$

#2a) $\sum \frac{(-1)^n (x-2)^n}{n\,2^n}$ ... ctr $= a = 2$ ... Radius of Conv ✓ Ratio Test

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n\,2^n}{(-1)^n (x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{n}{(n+1)2}(x-2)\right| \quad \text{HP}$$

$\leftarrow x \neq 2$

$$= \frac{1}{2}|x-2| < 1 \Rightarrow |x-2| < 2 \qquad R=2$$

Endpts:

$x - 2 = 2 \Rightarrow x = 4$ or $x - 2 = -2 \Rightarrow x = 0$

At $x=4$: $\sum \frac{(-1)^n (4-2)^n}{n(2)^n} = \sum \frac{(-1)^n 2^n}{n\,2^n} = \sum \frac{(-1)^n}{n}$

Alternating series: $a_n = 1/n > 0$

1. $\lim_{n\to\infty} 1/n = 0$ ✓

2. Decreasing? $\frac{1}{n+1} < \frac{1}{n}$ so $a_{n+1} < a_n$ $\therefore$ Decreases ✓

By Alternating series Test, the series converges @ $x=4$

At $x=0$: $\sum \frac{(-1)^n (0-2)^n}{n\,2^n} = \sum \frac{2^n}{n 2^n} = \sum 1/n$. diverges (p-series, $p=1$) So the series diverges @ $x=0$.

Interval: $(0, 4]$

2(b) $\sum \frac{3^n x^{n+1}}{(2n)!}$ — Radius of Convergence — ctr $= a = 0$. Ratio Test

$$\lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+2}}{(2n+2)!} \cdot \frac{(2n)!}{3^n x^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{3x}{(2n+1)(2n+2)} \right| = 0 < 1$$

$\leftarrow x \neq 0$ $\quad$ ∴ Converges for all x

$R = \infty$. Interval: $(-\infty, \infty)$

(c) $\sum_{k=1}^{\infty} k!\,(x+4)^k$ Ctr $a = -4$. Radius — Ratio Test

$$\lim_{n\to\infty} \left| \frac{(k+1)!\,(x+4)^{k+1}}{k!\,(x+4)^k} \right| = \lim_{k\to\infty} \left| (k+1)(x+4) \right| = \infty > 1.$$ $\boxed{\text{Converges only at } x = -4}$

$\leftarrow x \neq -4$

$R = 0$. No interval — converges only at $x = -4$

2(d) Bonus: $\sum \frac{k(x-10)^k}{3^k}$ ctr: $a = 10$ Radius of conv: Ratio Test

$$\lim_{k\to\infty} \left| \frac{(k+1)(x-10)^{k+1}}{3^{k+1}} \cdot \frac{3^k}{k(x-10)^k} \right| = \lim_{k\to\infty} \left| \frac{k+1}{k} \cdot \frac{(x-10)}{3} \right| = \frac{1}{3}|x-10| < 1$$

so $|x-10| < 3 = R$ $\boxed{R = 3}$. End pts $x - 10 = 3 \Rightarrow x = 13$

$x - 10 = -3 \Rightarrow x = 7$

At $x = 13$: $\sum_{k=0}^{\infty} \frac{k\,(13-10)^k}{3^k} = \sum k \frac{\cdot 3^k}{3^k} = \sum k$. Diverges by the nth term test $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty \neq 0$.

At $x = 7$ $\sum \frac{k(7-10)^k}{3^k} = \sum \frac{k(-3)^k}{3^k} = \sum (-1)^k k$ Diverges by nth term test: $\lim_{k\to\infty} (-1)^k k$ DNE. Interval: $(7, 13)$

3 Bonus $f(x) = \sqrt{x} = x^{1/2}$ find $p_3(x)$ ctr @ $x = 1$

$f(x) = (x)^{1/2}$	$f(1) = 1$
$f'(x) = 1/2\,(x)^{-1/2}$	$f'(1) = 1/2$
$f''(x) = -1/4\, x^{-3/2}$	$f''(1) = -1/4$
$f'''(x) = 3/8\, x^{-5/2}$	$f'''(1) = 3/8$

$$p_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(x)}{3!}(x-3)^3$$

$$p_3(x) = 1 + \tfrac{1}{2}(x-1) - 1/8\,(x-1)^2 + \tfrac{1}{16}(x-1)^3$$