Applications of Riemann Sums and the FTC: Net Distance Travelled

Suppose we know that the velocity of an object traveling along a line (think car on a straight highway) is given by a continuous function $v(t)$, where t represents time on the interval $[a, b]$. How might we determine the net distance the object has travelled? Well, we know that if the velocity were *constant*, then

$$\text{distance} = \text{rate} \times \text{time}.$$

Observe: Distance has been expressed as product, much the way we assumed earlier that the area of a rectangle could be expressed as a product:

$$\text{area of a rectangle} = \text{height} \times \text{base}.$$

We can extend this analogy to Riemann sums and area under curves. While the velocity is *not constant on long intervals* since the velocity is continuous it is *nearly constant on short time intervals*. So divide the time interval using a regular partition $\{t_0, t_1, t_2, \ldots, t_n\}$ of n subintervals of length Δt. Next, pick any point in the kth subinterval (we might as well choose the right-hand endpoint t_k for convenience) and evaluate the velocity $v(t_k)$ there. Then the distance traveled during the kth time interval approximated as

$$\text{distance} = \text{rate} \times \text{time} \approx v(t_k) \times \Delta t.$$

Since the net distance travelled is the sum of the distances traveled on each subinterval which is approximately

$$\text{Net Distance} \approx \sum_{k=1}^{n} v(t_k) \times \Delta t.$$

The approximation is improved by letting n get large and taking a limit.

$$\boxed{\text{Net Distance} = \lim_{n \to \infty} \sum_{k=1}^{n} v(t_k) \times \Delta t = \int_a^b v(t)\, dt.} \qquad (1.13)$$

Since v was assumed to be continuous, then we know that the limit exists and can be evaluated as a definite integral using antidifferentiation assuming we know an appropriate antiderivative. Finally, think about how we interpreted definite integrals geometrically: as (net) area under a curve. What we have just shown is that the net distance travelled over the time interval $[a, b]$ is just the *net area under the velocity curve*. That's not obvious at first.

What's your point? The key point here is that we were able to use a 'divide and conquer' process to determine the displacement. Let's list it as a series of steps.

- We subdivided the quantity into small bits,

- and we were able to approximate the each bit as a *product*.

- When we reassembled (summed) the bits, we found we had a Riemann sum.

- Once we had a Riemann sum we take a limit as the number of bits got large.

- The limit was a definite integral

- which we could evaluate easily (if we know an antiderivative) using the Fundamental Theorem of Calculus.

We will use this process repeatedly over the next few weeks.

☞ When first taking calculus it is easy to confuse the integration process (with its Riemann sums) with simple 'antidifferentiation.' While the First Fundamental Theorem connects these two, they are not the same thing. Most important, though, the determination of many quantities can be approximated (interpreted) as Riemann sums and hence evaluated as definite integrals *even though* it is not obvious at the outset that antidifferention should be involved. The Riemann sum part turns out to be critical.

Another Application of Definite Integrals: Average Value

Suppose that we want to know the average temperature for February 27, 2003 in Geneva How might we find it? Well, we could take the $n = 24$ hourly temperature recordings, add them together, and then divide by 24 might as we might do to find any average. Is the average 19.7 as listed in the table? What 'average' is that?
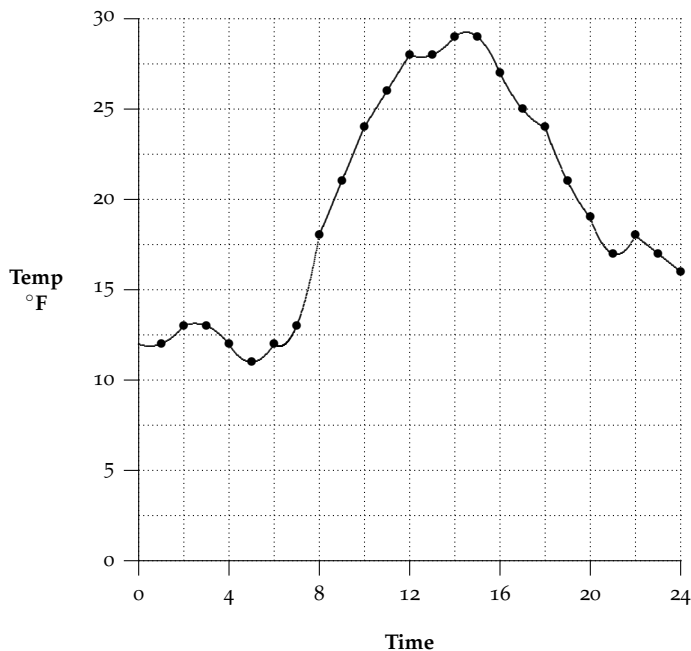
Time	Temp
1:00	12
2:00	13
3:00	13
4:00	12
5:00	11
6:00	12
7:00	13
8:00	18
9:00	21
10:00	24
11:00	26
12:00	28
13:00	28
14:00	29
15:00	29
16:00	27
17:00	25
18:00	24
19:00	21
20:00	19
21:00	17
22:00	18
23:00	17
24:00	16
Ave	19.7

Figure 1.49: A graph of the temperature on February 27, 2003 using the data to the right.

The average of 19.7 'privileges' those recordings made on the hour. We could get a better estimate if we recorded temperatures every half-hour, or every 5 minutes, or every minute, or perhaps every second. Sounds like a partition! The more recordings we use, the better the 'average.' Let's generalize the problem.

The Average Value Problem: Let f be a continuous function on the closed interval $[a, b]$. Find the average value of f on $[a, b]$.

> **SOLUTION.** Use the process we outlined. But how do we subdivide an average and make it product? As usual, start by dividing $[a, b]$ into n equal subintervals with partition points $\{x_0, x_1, \ldots, x_n\}$. Then, as we suggested above,
>
> $$\text{Average of } f \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \sum_{k=1}^{n} f(x_k) \cdot \frac{1}{n}. \qquad (1.14)$$
>
> The summation looks almost like a Riemann sum except we now have $\frac{1}{n}$ instead of Δx. But hold on!
>
> $$\Delta x = \frac{b - a}{n}$$
>
> so
>
> $$\frac{1}{n} = \frac{b - a}{n} \cdot \frac{1}{b - a} = \frac{\Delta x}{b - a}.$$
>
> Substituting this back in equation (1.14) gives
>
> $$\text{Average of } f \approx \sum_{k=1}^{n} f(x_k) \cdot \frac{\Delta x}{b - a} = \frac{1}{b - a} \sum_{k=1}^{n} f(x_k) \Delta x. \qquad (1.15)$$
>
> Now we do have a Riemann sum in (1.15). The best approximation occurs when we take a limit as the number of evaluation points $n \to \infty$. In other words
>
> $$\text{Average of } f = \lim_{n \to \infty} \frac{1}{b - a} \sum_{k=1}^{n} f(x_k) \Delta x = \frac{1}{b - a} \int_a^b f(x) \, dx. \qquad (1.16)$$