Math 131 Day 33

My Office Hours: M & W 12:30–2:00, Tu 2:30–4:00, & F 1:15-2:30 or by appointment. **Math Intern** Sun: 12–6pm; M 3-10pm; Tu 2–6, 7–10pm; W and Th: 5-10 pm in Lansing 310. Website: `http://math.hws.edu/~mitchell/Math131S13/index.html`.

Practice

Read **8.4** about the integral test. **Review all of 8.3** about series. Read the online notes.

1. **Vocabulary**: Make sure you know what each of the following terms means: series, partial sum, convergent (divergent) series, geometric series, nth term test for divergence, integral test, p-series.

2. Try page 567 #9, 15, 17, 19, 25, 27, 31, 33(use a substitution!), and 47.

Four Tests

1. The Geometric Series Test.

a) If $|r| < 1$, then the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ converges to $\dfrac{a}{1-r}$.

b) If $|r| \geq 1$, then the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges.

2. The nth term test for Divergence. If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (If $\lim\limits_{n\to\infty} a_n = 0$, this test is useless.)

3. The Integral Test. If $f(x)$ is a **positive**, **continuous**, and **decreasing** for $x \geq 1$ and $f(n) = a_n$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

4. The p-series Test. The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{cases}$

Hand in

Finish WeBWorK Day32 and start Day33. **The Day33 problems are EXCELLENT, especially on the integral test.**

1. Here are several series. Which of them can you say diverge by the nth term test for **divergence**? For which series is this test inconclusive? Explain. **Use appropriate mathematical language**. Pretend this is a test.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{2n+5}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ **c)** $\displaystyle\sum_{n=1}^{\infty} (1.1)^n$ **d)** $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{4}{n}\right)^n$

e) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}$ **f)** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

2. The integral test could be used to determine whether each of the following series converges or diverges. However, using the integral test is often a lot of work. For three of the series below it is possible to use one of the other tests (geometric series test or p-series test, there are some of each). Determine whether each series converges using the simplest method. Your answer should consist of a little 'argument' (a sentence or two) and any necessary calculations to show whether the series converges or diverges. **Use appropriate mathematical language**.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9+n^2}$ **e)** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+7n+10}$

Extra Credit for Extra Practice. Same instructions as above. Determine whether each converges.

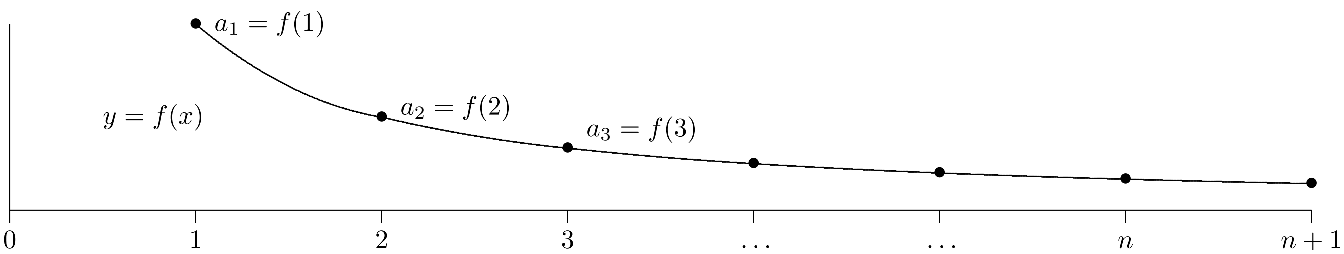
f) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n^3}}$ **g)** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ **h)** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{1.00001}}$ **i)** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

Series and Integrals: The Integral Test

The integral test is great! It combines a number of key concepts in the course: Riemann sums, improper integrals, sequences, and series. Yet it is a very intuitive result. Here's the idea. **Assume that:**

1) We have a series $\sum_{n=1}^{\infty} a_n$ where a_n is a function $f(n)$ defined on the positive integers.

2) Assume that the corresponding function $f(x)$ of the continuous variable x on the interval $[1, \infty)$ is **positive**, **continuous**, and **decreasing**. E.g., $\sum \frac{1}{n}$ where $f(x) = \frac{1}{x}$ is positive, continuous, and decreasing on $[1, \infty)$.



Each • indicates a point of the sequence $\{a_n\}$ with the graph of the corresponding function $f(x)$ for $x \geq 1$. Notice that $f(x)$ is positive, decreasing, and continuous.

We approximate the area under $f(x)$ on the interval $[1, n+1]$ by using both a **left**-hand Riemann sum, Left(n) and a **right**-hand Riemann sum, Right(n) with $\Delta x = \frac{(n+1)-1}{n} = 1$.

☞ Because f is decreasing, Left(n) is an ____________________ estimate for $\int_1^{n+1} f(x)\,dx$.

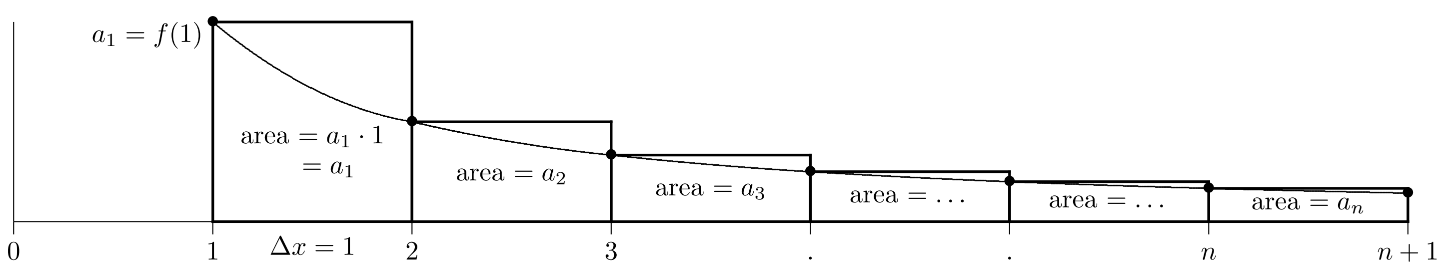
☞ Because f is decreasing, Right(n) is an ____________________ estimate for $\int_1^{n+1} f(x)\,dx$.

☞ So: Right(n) $\qquad \int_1^{n+1} f(x)\,dx \qquad$ Left(n).

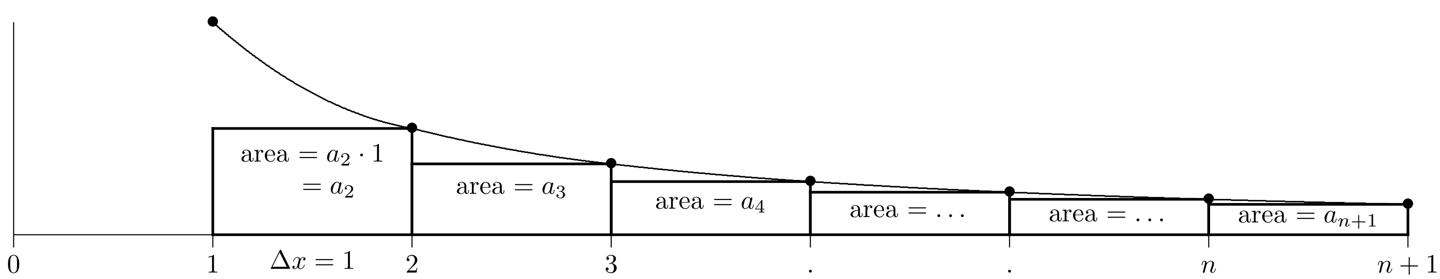
Study the graph of Left(n) below: Each rectangle has width $\Delta x = 1$ and height $f(k) = a_k$, so the area of the kth rectangle is just a_k. So the Riemann sum is

$$\text{Left}(n) = \sum_{k=1}^{n} a_k = S_n$$

That is, the left-hand Riemann sum is just the nth partial sum of the series. **Wow!**



What about Right(n)? Study the figure below and write out the sum. Right$(n) = \sum_{k=}$



We know that

$$\text{Right}(n) \leq \int_1^{n+1} f(x)\,dx \leq \text{Left}(n)$$

or

$$\sum_{k=2}^{n+1} a_k \le \int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k$$

Taking the limit as $n \to \infty$ we get the improper integral in the middle:

$$\sum_{k=2}^{\infty} a_k \le \int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \tag{1}$$

Now suppose that the improper integral diverges (goes to ∞). Since the full series is even bigger, the series must diverge, too.

☞ Take-home Message 1: If the improper integral **diverges** to infinity, so does the corresponding series!

On the other hand, if the series $\sum_{k=1}^{n} a_k$ diverges, so does series $\sum_{k=2}^{n} a_k$ since the first few terms of a series don't matter for convergence or divergence. But in equation (1), the improper integral $\int_1^{\infty} f(x)\,dx$ is bigger than $\sum_{k=2}^{n} a_k$, so the improper integral must diverge, too.

☞ Take-home Message 2: Therefore, if the series **diverges**, so does the improper integral!!

We can combine the two take-home messages into the following neat theorem.

Theorem: The Integral Test. Given $\sum_{n=1}^{\infty} a_n$ and a **positive**, **continuous**, and **decreasing** function $f(x)$ such that $f(n) = a_n$.

☞ Then either both $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ diverge or both converge.

Note 1: It is sufficient if $f(x)$ is positive and decreasing on some interval of the form $[a, \infty)$ where $a > 1$. It is the infinite tail of the the improper integral or the series that determines convergence or divergence, not the first few terms.

Note 2: If the integral converges, so does the series (though to a different value).

Math 131 Day 33: Practest 3

Part 1

For basic practice problems review Labs 9–11. Problems 1–4 apply our new techniques of integration to earlier applications.

1. a) Someone takes a maintenance medication: 48 mg once every 24 hr. Every 24 hr one-half of the drug is **eliminated** from the blood stream. Find the recurrence relation for the sequence $\{d_n\}$ where d_n is the amount of the drug in the bloodstream immediately after dose n.

b) Write out the first four terms of the sequence. Does the sequence appear to be monotonic?

c) Find the limit L of the sequence.

2. Find the average value of $f(x) = \frac{1}{x^2-9x+20}$ on the interval $[1, 3]$.

3. Find the volume of the infinitely long solid region generated when the area in the first quadrant enclosed by $y = \sqrt{\dfrac{8}{x^2+6x+5}}$, from $x = 0$ to $x = \infty$ is revolved around the x-axis. Use disks.

4. Find the volume of the solid region generated when the area enclosed by $y = \sqrt{\dfrac{8}{x^2+6x+5}}$, from $x = -1$ to $x = 0$ is revolved around the x-axis. Use disks.

5. a) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$ $y = 0$, and $x = 3$ and $x = 5$.

b) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$ $y = 0$, and $x = 5$ and $x = \infty$.

Part 2: Techniques

1. Try these; many are similar looking integrals. The first few are improper.

a) $\displaystyle\int_0^\infty \frac{4}{4+x^2}\,dx$ **b)** $\displaystyle\int_3^\infty \frac{4}{4-x^2}\,dx$ **c)** $\displaystyle\int_1^2 \frac{4}{4-x^2}\,dx$ **d)** $\displaystyle\int_3^\infty \frac{4x}{4-x^2}\,dx$

e) $\displaystyle\int \frac{4x+1}{x^2-5x+4}\,dx$ **f)** $\displaystyle\int \frac{4x+8}{x^2+4x+5}\,dx$

g) $\displaystyle\int \frac{-4x+4}{(x-2)^2x}\,dx$ **h)** $\displaystyle\int \frac{4x+1}{x^2-4}\,dx$ **i)** $\displaystyle\int \frac{4}{(4-x)^{2/3}}\,dx$

j) $\displaystyle\int \frac{8x+4}{x^3+x^2-2x}\,dx$ **k)** $\displaystyle\int_0^2 \frac{2x+6}{x^2+2x-8}\,dx$

l) $\displaystyle\int_4^\infty \frac{-3}{x^2-3x}$ **m)** $\displaystyle\int \frac{4x^2+8x+2}{x(x+1)^2}\,dx$ **n)** $\displaystyle\int \frac{4x}{(x+1)^3}\,dx$

2. Use L'Hopital's Rule if appropriate. (Answers **not** in order: 0, 0, 0, $\frac{1}{2}$, 1, 1 ln 2, 2, e, 3, 4, 5, -6, e^7.)

a) $\displaystyle\lim_{x\to1} \frac{x-1}{\ln x}$ **b)** $\displaystyle\lim_{x\to0} \frac{x^2+4x}{\sin 2x}$ **c)** $\displaystyle\lim_{x\to\infty} \frac{6x^2+3x-1}{2x^2+x}$

d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{e^x}$ **e)** $\displaystyle\lim_{x\to1} \frac{6x+10}{3x+1}$ **f)** $\displaystyle\lim_{x\to0} \frac{\tan 5x}{\arcsin x}$ **g)** $\displaystyle\lim_{x\to\infty} \left(1+\tfrac{7}{x}\right)^x$

h) $\displaystyle\lim_{x\to0} \frac{\cos 4x-\cos 2x}{x^2}$ **i)** $\displaystyle\lim_{x\to0} \frac{e^x-1-x}{x^2}$ **j)** $\displaystyle\lim_{x\to0^+} 2x\ln x$ **k)** $\displaystyle\lim_{x\to\infty} x^2e^{-x}$

l) $\displaystyle\lim_{x\to\infty} \ln(2x+9)-\ln(x+7)$ **m)** $\displaystyle\lim_{x\to0^+} (2x)^x$ **n)** $\displaystyle\lim_{x\to0^+} (1+x)^{1/x}$

3. Find the limits of these sequences. Use the key limits when possible (indicate when you do so). For the last part, use the derivative formula: $\frac{d}{dx}(a^x) = a^x \ln a$, when $a > 0$.

a) $\displaystyle\left\{\left(1+\frac{3}{n}\right)^n\right\}_{n=1}^\infty$ **b)** $\left\{\ln(2n^2+7)-\ln(5n^2+n)\right\}_{n=1}^\infty$ **c)** $\displaystyle\left\{\frac{2\ln(n+1)}{n^2}\right\}_{n=1}^\infty$

d) $\displaystyle\left\{\left(\frac{2}{3}\right)^n\right\}_{n=1}^\infty$ **e)** $\displaystyle\left\{\left(\frac{-3}{2}\right)^n\right\}_{n=1}^\infty$ **f)** $\displaystyle\left\{\frac{4n^2-3n+1}{5n^2+7}\right\}_{n=1}^\infty$

g) $\displaystyle\left\{\left(\frac{n^2}{3^n}\right)\right\}_{n=1}^\infty$

4. Find the sums of these series, if they exist. Note the starting indices!

a) $\displaystyle\sum_{n=0}^\infty 4\left(\frac{-2}{5}\right)^n$ **b)** $\displaystyle\sum_{n=0}^\infty 4\left(\frac{10}{9}\right)^n$ **c)** $\displaystyle\sum_{n=1}^\infty \frac{2}{n^2+n}$

d) $\displaystyle\sum_{n=0}^\infty \frac{6}{n^2+7n+12}$ **e)** $\displaystyle\sum_{n=3}^\infty \left(\frac{1}{2}\right)^n$ **f)** $\displaystyle\sum_{n=1}^\infty 3\left(\frac{-2}{3}\right)^n$

5. Determine whether the following series converge. First determine which test to use: nth term test, p-series test, integral test, geometric series test, or the comparison tests. Your final answer should consist of a little 'argument' (a sentence or two) and any necessary calculations. **Use appropriate mathematical language**.

a) $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt[3]{n^7}}$ **b)** $\displaystyle\sum_{n=1}^\infty \frac{e^n}{1+e^n}$ **c)** $\displaystyle\sum_{n=1}^\infty \frac{1}{16+9n^2}$ **d)** $\displaystyle\sum_{n=1}^\infty \frac{2^n+1}{n^2}$

e) $\displaystyle\sum_{n=1}^\infty \ln(3n+3)-\ln(6n+2)$ **f)** $\displaystyle\sum_{n=1}^\infty \frac{\sin\frac{1}{n}}{\frac{1}{n}}$ **g)** $\displaystyle\sum_{n=1}^\infty 2\left(\frac{-3}{11}\right)^n$ **h)** $\displaystyle\sum_{n=2}^\infty \frac{n}{\ln n}$

i) $\displaystyle\sum_{n=1}^\infty 6\left(\frac{5}{4}\right)^n$ **j)** $\displaystyle\sum_{n=0}^\infty \frac{3n^2}{n^3+1}$ **k)** $\displaystyle\sum_{n=1}^\infty \frac{(2n+1)!}{(2n-1)!}$ **l)** $\displaystyle\sum_{n=2}^\infty \frac{3}{n^2+5n+4}$

m) $\displaystyle\sum_{n=1}^\infty \frac{n^n}{n!}$ **n)** $\displaystyle\sum_{k=1}^\infty \frac{2k}{e^{k^2}}$ **o)** $\displaystyle\sum_{k=1}^\infty \frac{2}{k^2+4k+3}$ **p)** $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$