Math 131 Lab 1

1. 2-minute Review. These should be quick. Remember you can always check that an antiderivative is correct by taking its derivative.

   a) \( \int 2 + 6x - 9x^3 \, dx \)
   b) \( \int 4x^{-1} - e^{3x} \, dx \)
   c) \( \int \frac{\cos x}{2} \, dx \)
   d) \( \int c \, dx \)
   e) \( \int \sqrt{x} \, dx \)

2. Evaluate these limits (use L’Hopital’s Rule if appropriate). First classify the indeterminate type of each.

   a) \( \lim_{x \to 0} \frac{\arctan 4x}{\sin 2x} \)
   b) \( \lim_{x \to \infty} \frac{e^x - 1 - x}{x^2} \)
   c) \( \lim_{x \to 0^+} x \ln x \)
   d) \( \lim_{x \to \infty} x \ln x \)
   e) \( \lim_{x \to 0} \frac{2^x - 4x}{x + 5x^2} \)

3. Shortly we will need to figure out formulas for various sums. One of the simplest that you can figure out today is the sum of the first \( n \) integers. Let \( S_n = 1 + 2 + 3 + \cdots + (n - 1) + n \). For example \( S_4 = 1 + 2 + 3 + 4 = 10 \) (as any bowler would know). There a formula for \( S_n \) that C. F. Gauss (a very famous 19th century mathematician) figured out when he was 6. Here’s how: Write the summands forwards and backwards like so:

   \[
   S_n = 1 + 2 + \cdots + (n - 1) + n
   \]

   \[
   + S_n = n + (n - 1) + \cdots + 2 + 1
   \]

   \[
   2S_n = \]

   a) Now add each column. What do you get as the total for each? How many times?
   b) So what is the formula for \( 2S_n \)? Now solve for \( S_n \).
   c) Use your formula to check that \( S_4 = 10 \). Now use it to determine \( S_{10} \) and \( S_{100} \). Quick!
   d) Suppose you wanted to sum the even integers only. Let \( T_n \) be the sum of the first \( n \) even integers: \( T_n = 2 + 4 + \cdots + 2n \). What is the formula for \( T_n \)?

4. Find the general antiderivatives of these somewhat more interesting functions. Remember you can check your answers by differentiating. For (e), divide first. Remember ‘+c’.

   a) \( \int 6x^{-1/2} - 4 \sec^2 x \, dx \)
   b) \( \int \sqrt{x^2} \, dx \)
   c) \( \int \frac{4}{7 \sqrt{x^4}} \, dx \)
   d) \( \int \sin 2x \, dx \)
   e) \( \int \frac{9x^3 - 2x + x^{1/2}}{x^2} \, dx \)
   f) \( \int \frac{5}{\sqrt{1 - x^2}} \, dx \)
   g) \( \int \sec(3x) \tan(3x) \, dx \)
   h) \( \int \cos^2 x + \sin^2 x \, dx \)
   EZ!

5. a) From Calculus I, the Extreme Value Theorem says that any continuous function on a closed interval \([a, b]\) has both maximum and a minimum value on the interval. The two graphs below are both continuous. For each of the closed subintervals listed, estimate the max and min values of the functions \( f \) and \( g \).

   b) For the function \( f \), draw the three rectangles whose bases are the intervals along the \( x \)-axis and whose heights are the min values of \( f \) on each interval. What is the sum of their areas? Is this sum an over or underestimate the area under the curve? Why?
   c) For the function \( g \), draw the three rectangles whose bases are the intervals along the \( x \)-axis and whose heights are the MAX values of \( g \) on each interval. What is the sum of their areas? Over or underestimate? Why?

6. Determine the particular function \( f(x) \) such that \( f'(x) = 6x^2 + 1 \) and \( f(1) = 4 \). First find the general antiderivative, then evaluate \( c \) by using the given value. There should be no unknown constant term “\( c \)” in your final answer.

7. Suppose that \( f''(x) = 2 \cos x \). If \( f'(0) = 1 \) and \( f(0) = 2 \), what is the function \( f(x) \)? Hint: Do antidifferentiation twice.
8. A sea otter ingests a pollutant and immediately begins to excrete it at a rate of
\[
\frac{dA}{dt} = \frac{k}{1 + t^2},
\]
where \(A(t)\) is the amount of the pollutant (in mg) remaining after \(t\) days and \(k\) is an unknown constant. If the initial amount of the pollutant ingested is \(A(0) = 20\) mg and a day later there is \(A(1) = 15\) mg left, find the function \(A(t)\). How much of the pollutant remains after 5 days?

9. **Sigma Notation.** You may have seen sigma notation before. It is used to indicate the summation of a number of terms that follow some pattern.

This tells us to
end with \(i = n\)
\[
\downarrow
\]
This tells us to sum → \[\sum_{i=m}^{n} a_i \] ← what to sum. Often
\[
\uparrow
\]
This tells us to
start with \(i = m\)

For example, \[\sum_{i=1}^{5} 2i = 2(1) + 2(2) + 2(3) + 2(4) + 2(5)\]. Translate each of the following:

a) \[\sum_{i=1}^{4} i^2\]  
b) \[\sum_{j=3}^{5} \cos(j\pi x)\]  
c) \[\sum_{j=0}^{3} j^3 + 2j\]

Now write each of the following sums using sigma notation.

d) \[4 + 6 + 8 + 10 + 12\]  
e) \[3 + 9 + 27 + 81 + 243\]  
f) \[−1 + 1 − 1 + 1 − 1 + 1\]  
g) \[0 + 1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{20}\]

10. a) Find the derivative of \(y = x \ln x - x\).

b) Determine \(\int \ln x \, dx\). Hint: Look up.

11. Sometimes when we have no formula for a function we are forced to do graphical antidifferentiation. Let \(F(x)\) be the antiderivative of \(f(x)\) on \([-3, 4]\), where \(f\) is the function graphed on the left below. Since \(F\) is an antiderivative of \(f\), then \(F' = f\). Use this relationship to answer the following questions.

a) Where is \(F'\) positive? Negative? Use \(F'\) to determine the interval(s) where \(F\) increasing. Decreasing.

b) At what point(s), if any, does \(F\) have a local max? Min?

c) Determine where \(F''\) is positive and negative. On what interval(s) is \(F\) concave up? Down?

d) Does \(F\) have any points of inflection?

e) Assume \(F\) passes through the point \((-3, 1)\) indicated with a •; draw a potential graph of \(F\).

f) Assume, instead, that \(F\) passes through \((-3, -1)\) indicated by a ◦; draw a graph of \(F\).

g) What is the relationship between the two graphs you’ve drawn?

12. If time allows: Repeat Problem 11 for the graph on the right above.
Answers to Math 131 Lab 1

1. The antiderivatives are:
   a) \( 2x + 3x^2 - \frac{9}{4} x^4 + c \)  
   b) \( 4 \ln |x| - \frac{1}{2} e^{3x} + c \)
   c) \( \frac{1}{2} \sin x + c \)  
   d) \( cx + d \)  
   e) \( \frac{3}{4} x^{4/3} + c \)

2. a) \( \lim_0^0 \frac{\arctan 4x}{\sin 2x} = \lim_0^0 \frac{\frac{4}{1 + 16x^2}}{2\cos 2x} = \frac{\frac{4}{2}}{2} = 2. \)
   b) \( \lim_0^0 \frac{e^x - 1 - x}{x^2} = \lim_0^0 \frac{e^x - 1 - \frac{1}{2} x}{2x} = \lim_0^0 \frac{e^x - \frac{1}{2} x}{2} = 1. \)
   c) \( \lim_0^\infty x \ln x = \lim_0^\infty \frac{\ln x}{\frac{1}{x}} = \lim_0^\infty \frac{1}{\frac{1}{x}} = \lim_0^\infty x = \lim_0^\infty -x = 0. \)
   d) \( \lim_0^\infty \frac{x \ln x}{x^2 + 1} = \lim_0^\infty \frac{1 + \ln x}{2x} = \lim_0^\infty \frac{\frac{1}{2} x}{2} = 0 = 0. \)
   e) \( \lim_0^0 \frac{2x - 4x}{x^2 + 1} = \lim_0^0 \frac{2x}{1 + 10x} = \lim_0^0 \frac{2}{10} = \frac{2}{10} = -\ln 2. \)

3. We have:
   \[
   S_n = \frac{1}{n} + \frac{2}{n} + \cdots + \frac{(n-1)}{n} \quad (n+1) + \frac{1}{n+1} + \cdots + \frac{(n+1)}{n+1}
   \]
   a) We get \( n+1 \) a total of \( n \) times.
   b) So \( 2S_n = n(n+1) \), or \( S_n = \frac{n(n+1)}{2} \).
   c) \( S_4 = \frac{4(5)}{2} = 10 \). \( S_{10} = \frac{10(11)}{2} = 55 \) and \( S_{100} = \frac{100(101)}{2} = 5050 \).
   d) \( T_n = 2+4+\cdots+2n = 2(1+2+\cdots+n) = 2S_n = n(n+1) \).

4. The antiderivatives are:
   a) \( 12x^{3/2} - 4 \tan x + c \)  
   b) \( \frac{5}{7} x^{7/5} + c \)  
   c) \( \int \frac{4}{7} x^{-4/3} \, dx = -\frac{12}{7} x^{-1/3} + c \)
   d) \( -\cos(2x) + c \)
   e) \( \int 9x - 2x^{-1} + x^{-3/2} \, dx = \frac{9}{2} x^2 - 2 \ln |x| - 2x^{-1/2} + c \)
   f) \( 5 \arcsin x + c \)  
   g) \( \frac{\sec(3x)}{3} + c \)
   h) \( x + c \)

For part (h), remember that \( \cos^2 x + \sin^2 x = 1 \).

5. a) Maxs are denoted with \( \bullet \) and mins with \( \times \). Why are some points marked as both a max and a min?

b) Area sum: \( 3 \times 0.5 + 2 \times 4.5 + 3 \times 2 = 16.5 \). Underestimate. The rectangles are below the curve.

c) Area sum: \( 3 \times (-3) + 2 \times (-2.2) + 3 \times 2 = 16.5 - 7.4 \). Overestimate. The rectangles are always above the curve.

6. If \( f'(x) = 6x^2 + 1 \), then \( f(x) = 2x^3 + x + c \). But \( f(1) = 4 = 2 + 1 + c \Rightarrow c = 1 \). So \( f(x) = 2x^3 + x + 1 \).

7. If \( f''(x) = 2 \cos x \), then \( f'(x) = \int 2 \cos x \, dx = 2 \sin x + c \). But \( f'(0) = 2(0) + c = 1 \Rightarrow c = 1 \). So \( f'(x) = 2 \sin x + 1 \) so \( f(x) = -2 \cos x + x + c \). But \( f(0) = -2 + 0 + c = 2 \), so \( c = 4 \). Now \( f(x) = -2 \cos x + x + 4 \).
8. Given \( \frac{dA}{dt} = \frac{k}{1+t^2} \), \( A(0) = 20 \), and \( A(1) = 15 \). So

\[
A(t) = \int A'(t) \, dt = \int \frac{k}{1+t^2} \, dt = k \arctan(t) + c.
\]

\( A(0) = 20 = 0 + c \), so \( c = 20 \). Therefore, \( A(t) = k \arctan(t) + 20 \). Then

\[
A(1) = 15 = k \arctan(1) + 20 = k\frac{\pi}{4} + 20 \rightarrow -5 = k\frac{\pi}{4} \rightarrow k = -\frac{20}{\pi}.
\]

Therefore, \( A(t) = -\frac{20}{\pi} \arctan t + 20 \), so \( A(5) = -\frac{20}{\pi} \arctan(5) + 20 \approx 11.26 \) mg.

9. Sigma Notation.

\( a) \sum_{i=1}^{4} i^2 = 1 + 4 + 9 + 16 = 30 \quad b) \sum_{j=3}^{5} \cos(j\pi x) = \cos(3\pi x) + \cos(4\pi x) + \cos(5\pi x) = -1 + 1 - 1 = -1 \)

\( c) \sum_{j=0}^{3} j^3 + 2j = 0 + 3 + 12 + 33 = 38 \)

\( d) \sum_{j=2}^{6} 2j \quad e) \sum_{j=1}^{5} 3^j \quad f) \sum_{j=1}^{6} (-1)^j \quad g) \sum_{j=0}^{20} \sqrt{j} \)

10. a) \( y' = \ln x + \frac{x}{2} - 1 = \ln x \). b) Looking at part (a): \( \int \ln x \, dx = x \ln x - x + c \).

11. a) Use that \( F' = f \) and \( F'' = f' \). \( F \) increasing means \( F' = f > 0 \): on \((-3, 0) \) and \((3, 4)\). \( F \) decreasing means \( F' = f < 0 \): on \((0, 3)\).

b) \( F \) has a local max means \( F' = f \) changes from + to -: at \( x = 0 \). \( F \) has a local min means \( F' = f \) changes from - to +: at \( x = 3 \).

c) \( F \) concave up means \( F'' = f' > 0 \), i.e., \( f \) is increasing: \((-3, -1.5) \) and \((1.5, 4)\). \( F \) concave down means \( F'' = f' < 0 \), i.e., \( f \) is decreasing: \((-1.5, 1.5)\).

d) \( F \) has a points of inflection when \( F'' = f' \) changes sign, i.e., when \( f \) changes from increasing to decreasing or vice versa: at \( x = \pm 1.5 \).

g) The two graphs are parallel (differ only by a constant).

12. This is a future HW problem.