# Math 131 Lab 14: Series

**1.** Determine whether these arguments are correct. If not, correct them.

**a)** Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n(2n^4+1)}{6n^9+2}$ converges absolutely, conditionally, or diverges. ARGUMENT: Use the alternating series test with $a_n = \frac{2n^4+1}{6n^9+2} > 0$. Check the two conditions: (i): $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^4+1}{6n^9+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^4}{6n^9} = \lim_{n\to\infty} \frac{2}{6n^5} = 0.$✓(ii): Decreasing? Take the derivative! (You can't just say the bottom gets bigger since the top gets bigger, too!) $f(x) = \frac{2x^4+1}{6x^9+2}$ so

$$f'(x) = \frac{8x^3(6x^9+2) - (2x^4+1)54x^8}{(6x^9+1)^2} = \frac{-60x^{12} - 54x^8 + 16x^3}{(6x^9+1)^2} < 0$$

so $f(x)$ and $a_n$ are decreasing.✓So by the Alternating Series Test the series converges conditionally.

**b)** Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{6n+2}$ converges absolutely, conditionally, or diverges. ARGUMENT: Notice $\sum_{n=1}^{\infty} \left|\frac{(-1)^n(2n+1)}{6n+2}\right| = \sum_{n=1}^{\infty} \frac{2n+1}{6n+2} = \lim_{n\to\infty} \frac{2n+1}{6n+2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n}{6n} = \frac{1}{3}$. So the series converges absolutely.

**c)** Determine the radius and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{x^n}{3n^2+1}$. ARGUMENT: We know that the series converges at its center $a = 0$. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{3(n+1)^2+1} \cdot \frac{3n^2+1}{x^n}\right| = \lim_{n\to\infty} \left|\frac{3n^2+1}{3n^2+6n+4} \cdot x\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{3n^2}{3n^2} \cdot x\right| = |x|.$$

By the ratio test, the series *converges* if $|x| < 1$ and *diverges* when $|x| > 1$. The radius of convergence is $R = 1$. Check the endpoints $a - R = 0 - 1 = -1$ and $a + R = 0 + 1 = 1$. **For** $x = 1$**:** We get $\sum_{n=0}^{\infty} \frac{1}{3n^2+1}$. Since $0 < \frac{1}{3n^2+1} < \frac{1}{n^2}$ and since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges ($p$-series, $p = 2 > 1$), then by direct comparison $\sum_{n=0}^{\infty} \frac{1}{3n^2+1}$ converges. **For** $x = -1$**:** We get $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^2+1}$. However, we just saw that $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{3n^2+1}\right| = \sum_{n=0}^{\infty} \frac{1}{3n^2+1}$ converges. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^2+1}$ converges by the absolute convergence test. The interval of convergence is $[-1, 1]$ and includes both endpoints.

**2.** Determine whether these series converge absolutely, conditionally, or not at all. (Hint: For a couple of these consider the Ratio Test Extension for absolute convergence/divergence)

**a)** $\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{3n^2+2}$ **b)** $\sum_{n=1}^{\infty} \frac{(-1)^n n2^n}{3^{n+1}}$ **c)** $\sum_{n=1}^{\infty} \frac{n!}{(-3)^n}$

**3.** Find the radius $R$ and interval of convergence for each of these series. Remember the endpoints.

**a)** $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$ **b)** $\sum_{n=1}^{\infty} \frac{(-1)^n n! x^n}{4n}$ **c)** $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$

**d)** $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n+1}$ **e)** $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{3^{2n}}$ **f)** $\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$

**4. a)** Find the Taylor polynomial $p_4(x)$ for the function $f(x) = e^{-2x}$ centered at $a = 0$.

**b)** What is formula for the infinite degree Taylor polynomial $p_\infty(x)$. Express your answer as a series.

**c)** What is the radius of convergence of this series for $p_\infty(x)$?

**5.** Find the radius of convergence $R$ for $\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$ and (Bonus!) for $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$.

## Brief Answers

**1. a)** Wrong. (b) Wrong. (c) Correct. This is a model of how I want your answers in #3 to be!

**2.** Conditional, Absolute, Diverge.

**3.** Lots of justification (words) required!

**a)** $[-1/2, 1/2]$ **b)** $\{0\}$ **c)** $(-3, 3)$ **d)** $[3, 5)$ **e)** $(-8, 10)$ **f)** $(-\infty, \infty)$

**4. a)** $p_4(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4$.

**b)** $p_\infty(x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k$.

**c)** The interval of convergence is $(-\infty, \infty)$.

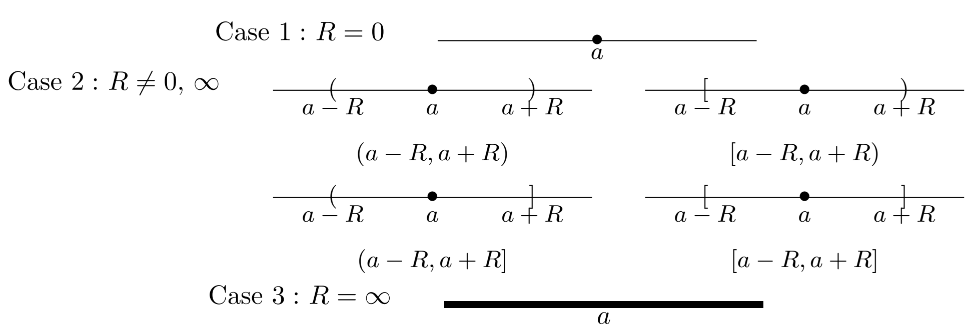
**5. a)** $R = 2$.

## Background

**1. Power Series.** For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at $a$, precisely one of the following is true.

**a)** The series converges only at $x = a$.
**b)** There is a real number $R > 0$ so that the series converges absolutely for $|x - a| < R$ and diverges for $|x - a| > R$.
**c)** The series converges for all $x$.

NOTE: In case (b) the power series may converge at both endpoints, either endpoint, or neither endpoint. You have to check the convergence at the endpoints separately. Here's what the intervals of convergence can look like:




**2. The Ratio Test Extension.** Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **non-zero** terms and let $r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

1. If $r < 1$, then the series $\sum a_n$ converges *absolutely.*
2. If $r > 1$ (including $\infty$), then the series $\sum a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

**3. The Alternating Series Test.** Assume $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the follow two conditions hold:

**a)** $\lim_{n\to\infty} a_n = 0$
**b)** $a_{n+1} \le a_n$ for all $n$ (i.e., $a_n$ is decreasing which can also be tested by showing $f'(x) < 0$).

**4. Absolute Convergence Test.** If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

## Lab 14 Answers

**1. a)** Wrong. The student never checked for absolute convergence. ARGUMENT: First we check absolute convergence. $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n^4+1)}{6n^9+2}\right| = \sum_{n=1}^{\infty}\frac{2n^4+1}{6n^9+2}$. Notice that $\frac{2n^4+1}{6n^9+2} \approx \frac{1}{n^5}$. So let's use the limit comparison test. The terms of the series are positive and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^4+1}{6n^9+2}\cdot\frac{n^5}{1} = \lim_{n\to\infty}\frac{2n^9+n^5}{6n^9+2} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n^9}{6n^9} = \frac{1}{3}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges ($p$-series with $p=5>1$), then $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n^4+1)}{6n^9+2}\right|$ converges by the limit comparison test. So the series converges absolutely.

**b)** Wrong. Puleeze never do this! You might start with the alternating series test with $a_n = \frac{2n+1}{6n+2} \neq 0$. Check the two conditions (i): $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n+1}{6n+2} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n}{6n} = \frac{1}{3}$. So the series diverges by the $n$th term test (not the alternating series test). It's the $n$th term test.

**c)** Correct. This is what I want your answers in #3 to be!

**2. a)** $\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{3n^2+2}$. The series is similar to $\sum_{n=1}^{\infty}\frac{1}{n}$, so it probably will not converge absolutely. ARGUMENT: Use the alternating series test with $a_n = \frac{2n+1}{3n^2+2} > 0$. Check the two conditions: (i): $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n+1}{3n^2+2} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n}{3n^2} = \lim_{n\to\infty}\frac{2}{3n} = 0.\checkmark$ (ii): Decreasing? Take the derivative. (You can't just say the bottom gets bigger since the top gets bigger, too!) $f(x) = \frac{2x+1}{3x^2+2}$ so

$$f'(x) = \frac{2(3x^2+2)-(2x+1)6x}{(3x^2+2)^2} = \frac{-6x^2-6x^8+4}{(3x^2+2)^2} < 0\checkmark \qquad (x \geq 1)$$

so $f(x)$ and $a_n$ are decreasing. By the Alternating Series Test the series converges. Now check absolute convergence. $\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n+1)}{3n^2+2}\right| = \sum_{n=1}^{\infty}\frac{2n+1}{3n^2+2}$. Compare to $\sum_{n=1}^{\infty}\frac{1}{n}$. Notice both $\frac{1}{n} > 0$ and $\frac{2n+1}{3n^2+2} > 0$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+1}{3n^2+2}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{2n^2+n}{3n^2+2} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n^2}{3n^2} = \frac{2}{3} > 0.$$

Since $\sum_{n=0}^{\infty}\frac{1}{n}$ diverges by the $p$-series test ($p=1$), then by the limit comparison test $\sum_{n=0}^{\infty}\frac{2n+1}{3n^2+1}$ also diverges. So the series is conditionally convergent.

**b)** $\sum_{n=1}^{\infty}\frac{(-1)^n n2^n}{3^{n+1}}$. Because of the $n$th powers, try the ratio test first (testing for absolute convergence) with $a_n = \frac{n2^n}{3^{n+1}} \neq 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)2^{n+1}}{3^{n+2}}\cdot\frac{3^{n+1}}{(-1)^n n2^n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{3n}\right| \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n}{3n} = \frac{2}{3} < 1.$$

By the ratio test the series converges absolutely.

**c)** $\sum_{n=1}^{\infty}\frac{n!}{(-3)^n}$. Because of the $n$th power and factorial, try the ratio test first (testing for absolute convergence) with $a_n = \frac{n!}{(-3)^n} \neq 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(-3)^{n+1}}\cdot\frac{(-3)^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \infty > 1.$$

By the ratio test the series diverges.

**3. a)** $\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2}$. ARGUMENT: The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n}\right| = \lim_{n\to\infty} \left|\frac{n^2 2x}{(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{n^2}{n^2+2n+1} \cdot 2x\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{n^2}{n^2} \cdot 2x\right| = 2|x|.$$

By the ratio test, the series *converges* if $2|x| < 1 \iff |x| < \frac{1}{2}$ and *diverges* when $|x| > \frac{1}{2}$. The radius of convergence is $R = \frac{1}{2}$. Check the endpoints $-\frac{1}{2}$ and $\frac{1}{2}$? **For** $x = \frac{1}{2}$**:** We get

$$\sum_{n=0}^{\infty} \frac{(2(\frac{1}{2}))^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

which converges by the $p$-series test ($p = 2 > 1$). **For** $x = -\frac{1}{2}$**:** We get

$$\sum_{n=0}^{\infty} \frac{(2(-\frac{1}{2}))^n}{n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

But this series converges absolutely (we just did it). The interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

**b)** $\sum_{n=0}^{\infty} \frac{(-1)^n n! x^n}{4n}$. ARGUMENT: The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)! x^{n+1}}{4(n+1)} \cdot \frac{4n}{(-1)^n n! x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)n}{n+1} \cdot x\right| = \lim_{n\to\infty} |nx| = \infty > 1.$$

By the ratio test, the series *diverges* when $|x| > 0$. The radius of convergence is $R = 0$. The series only converges at $x = 0$.

**c)** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$. The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)}}{9^{n+1}} \cdot \frac{9^n}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{9}\right|.$$

By the ratio test, the series *converges* if $|\frac{x^2}{9}| < 1 \iff |x^2| < 9 \iff |x| < 3$ and *diverges* when $|x| > 3$. The radius of convergence is $R = 3$. Check the endpoints $a - R = 0 - 3 = -3$ and $a + R = 0 + 3 = 3$. **For** $x = 3$**:** We get

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{9^n} = \sum_{n=0}^{\infty} (-1)^n$$

which diverges by the geometric series test ($|r| = 1$). The same is true **for** $x = -3$**:** Again we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} (-1)^n.$$

The interval of convergence is $(-3, 3)$.

**d)** $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n+1}$. ARGUMENT: The series converges at its center $a = 4$. Apply the ratio test. For any $x \neq 4$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(x-4)^{n+1}}{n+2} \cdot \frac{n+1}{(x-4)^n}\right| = \lim_{n\to\infty} \left|\frac{n+2}{n+1} \cdot (x-4)\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{n}{n} \cdot (x-4)\right| = |x-4|.$$

By the ratio test, the series *converges* if $|x-4| < 1$ and *diverges* when $|x-4| > 1$. The radius of convergence is $R = 1$. Check the endpoints: $a - R = 4 - 1 = 3$ and $a + R = 4 + 1 = 5$. **For** $x = 3$**:** We get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

Use the alternating series test with $a_n = \frac{1}{n+1} \neq 0$. Check the two conditions. (1) $\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$✓and (2) $a_{n+1} \leq a_n$ since $\frac{1}{n+2} < \frac{1}{n+1}$ (decreasing).✓So the series converges at $x = 3$ by the alternating series test. **For** $x = 5$**:** We get

$$\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

Use the limit comparison test with $\sum_{n=0}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} 1.$$

Since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges by the $p$-series test ($p = 1$), then by the limit comparison test $\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges. The interval of convergence is $[3, 5)$.

**e)** $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-1)^n}{3^{2n}}$. The series converges at its center $a = 1$. Apply the ratio test. For any $x \neq 1$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-1)^{n+1}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{n(x-1)^n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{3^2 n} \cdot (x-1)\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left|\frac{n}{9n} \cdot (x-1)\right| = \left|\frac{x-1}{9}\right|.$$

By the ratio test, the series *converges* if $\left|\frac{(x-1)}{9}\right| < 1 \iff |x-1| < 9$ and *diverges* when $|x-1| > 9$. The radius of convergence is $R = 9$. What about the endpoints $a - R = 1 - 9 = -8$ and $a + R = 1 + 9 = 10$? **For** $x = -8$**:** We get

$$\sum_{n=1}^{\infty} \frac{n(1+8)^{2n}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{n(-9)^n}{9^n} = \sum_{n=1}^{\infty} (-1)^n n.$$

But $\lim\limits_{n\to\infty} (-1)^n n$ DNE, so by the $n$th term test the series diverges. **For** $x = 10$**:** We get

$$\sum_{n=1}^{\infty} \frac{n(10-1)^n}{9^n} = \sum_{n=1}^{\infty} n$$

and so the series diverges by the $n$th term test again. The interval of convergence is $(-8, 10)$.

**f)** $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$. The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{(2n+1)(2n+2)} \cdot x\right| = \lim_{n\to\infty} \left|\frac{1}{4n+2} \cdot x\right| = 0 < 1.$$

By the ratio test, the series for all $x$ The interval of convergence is $(-\infty, \infty)$.

**4. a)** Find the Taylor polynomial $p_4(x)$ for the function $f(x) = e^{-2x}$ centered at $a = 0$.

**b)** What is formula for the infinite degree Taylor polynomial $p_\infty(x)$. Express your answer as a series.

**c)** What is the radius of convergence of this series for $p_\infty(x)$?

**5. a)** The formula for $p_n(x)$ with center $a = 0$ is given by

$$p_n(x) = f(0) + f'(0) \cdot (x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n.$$

So we need to calculate the derivatives of $f(x) = e^{-2x}$ and then evaluate them at 0. Well,

$$\begin{aligned} f(x) &= e^{-2x} & f(0) &= 1 = (-2)^0 \\ f'(x) &= -2e^{-2x} & f'(0) &= -2 = (-2)^1 \\ f''(x) &= (-2)^2 e^{-2x} & f''(0) &= (-2)^2 \\ f'''(x) &= (-2)^3 e^{-2x} & f''(0) &= (-2)^3 \\ f'''(x) &= (-2)^4 e^{-2x} & f''(0) &= (-2)^4 \\ &\vdots & & \\ f^{(k)}(x) &= (-2)^k e^{-2x} & f'^{(k)}(0) &= (-2)^k \end{aligned}$$

$$p_4(x) = 1 - 2(x-0) + \frac{4}{2!}(x-0)^2 - \frac{8}{3!}(x-0)^3 + \frac{16}{4!}(x-0)^4) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

**b)** $p_\infty(x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k$.

**c)** $\sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k$. The series converges at its center $a = 0$. Apply the ratio test. For any $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \cdot x \right| = 0 < 1.$$

By the ratio test, the series for all $x$ The interval of convergence is $(-\infty, \infty)$.

**6. a)** $\sum_{n=1}^{\infty} \frac{n! x^n}{1\cdot 3\cdot 5\cdots(2n+1)}$. Use the ratio test.

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{1\cdot 3\cdot 5\cdots(2n+3)} \cdot \frac{1\cdot 3\cdot 5\cdots(2n+1)}{n! x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{2n+3} \right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \left| \frac{nx}{2n} \right| = \lim_{n\to\infty} \left| \frac{x}{2} \right|.$$

By the ratio test, the series *converges* if $|\frac{x}{2}| < 1 \iff |x| < 2$ and *diverges* when $|x| > 2$. The radius of convergence is $R = 2$.

**b)** $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$ Use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n x^n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x}{(n+1) n^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^n x}{n^n} \right| \\
&= \lim_{n\to\infty} \left| \left( \frac{n+1}{n} \right)^n \cdot x \right| = \lim_{n\to\infty} \left| \left( 1 + \frac{1}{n} \right)^n \cdot x \right| = |ex|.
\end{aligned}$$

By the ratio test, the series *converges* if $|ex| < 1 \iff |x| < \frac{1}{e}$. The radius of convergence is $R = \frac{1}{e}$.