Math 131 Lab 09

We have nearly completed our "techniques" of integration phase of the course. These problems primarily review triangle substitution and partial fractions, but other techniques are also used. Do the problems with ☞ first; come back to the others later or after lab.

0. Reference: Key Identities. Double angle and reduction formulæ.

a) $\sin(2\theta) = 2\sin\theta\cos\theta$ **b)** $\displaystyle\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

1. ☞ Determine $\displaystyle\int \frac{1}{9x^2\sqrt{1-9x^2}}\,dx$. Use triangles. If (when) you get stuck with the integration, go back and switch the x-side of the triangle. This is one of those few cases where the other choice is better.

2. ☞ the right triangle associated with each of these square roots and label the sides. For each, solve for u, du, and the given square root in terms of an angle θ.

a) $\sqrt{u^2-a^2}$ **b)** $\sqrt{a^2-u^2}$ **c)** $\sqrt{u^2+a^2}$

3. Determine these integrals. *Caution:* A variety of techniques are required. Where necessary, make use of the triangles that you just drew.

a) ☞ $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$ **b)** $\displaystyle\int \frac{1}{(36+x^2)^{3/2}}\,dx$ **c)** $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$

d) ☞ $\displaystyle\int x\sqrt{4-x^2}\,dx$ **e)** $\displaystyle\int \frac{4}{4-x^2}\,dx$ **f)** ☞ $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$

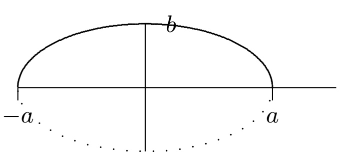
4. Try these rational function integrals.

a) ☞ $\displaystyle\int \frac{x+7}{x^2+2x-3}\,dx$ **b)** $\displaystyle\int \frac{2x}{8-2x-x^2}\,dx$ **c)** ☞ $\displaystyle\int \frac{8x-2}{x^3+x^2-2x}\,dx$

d) ☞ $\displaystyle\int \frac{x+4}{x^2+8x+2}\,dx$ **e)** $\displaystyle\int \frac{x+1}{x^2-4}\,dx$

5. Evaluate these definite integrals.

a) ☞ $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$ **b)** $\displaystyle\int_0^1 \frac{x^3}{1+x^2}\,dx$

6. 

☞ Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Hint: Find the area of the upper semi-ellipse. $y = \frac{b}{a}\sqrt{a^2-x^2}$.

7. This is similar to an extra credit problem on WeBWorK from Monday.

a) Determine $\int \sec^3 x\,dx$ by reduction.

b) ☞ Find the arc length of the parabola $y = f(x) = 3 + 2x^2$ on the interval $[0, \sqrt{3}]$.

8. a) Find the area of the region R between the curve $y = \frac{1}{6+x-x^2}$ and the x-axis on the interval $[0, 2]$. Use partial fractions.

b) Extra Credit. Hand in by Friday at class. Now suppose that R is rotated around the y-axis. Find the resulting volume by the shell method.

9. ☞ For this octet of problems, **just decide which technique** is most appropriate and give the initial step to do the integration. (E.g., give the substitution or parts. etc.)

a) $\displaystyle\int \frac{x}{\sqrt{4-x}}\,dx$ **b)** $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$ **c)** $\displaystyle\int \frac{4}{4-x^2}\,dx$ **d)** $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

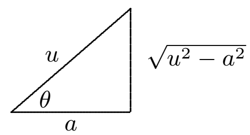
e) $\displaystyle\int \frac{4}{4+x^2}\,dx$ **f)** $\displaystyle\int \frac{4x}{4-x^2}\,dx$ **g)** $\displaystyle\int \frac{4x^3}{4-x^2}\,dx$ **h)** $\displaystyle\int \frac{x-1}{x^2+4x+3}\,dx$

Some Answers

Caution: Watch for typos. First to find a particular mistake gets Extra Credit.

1. $-\frac{\sqrt{1-9x^2}}{9x}+c$

2. a) In this case, u must correspond to the hypotenuse of the right triangle. (Why?) We have our choice of how to label the legs, one side a and the other $\sqrt{u^2-a^2}$. With the selection made below, $u=a\sec\theta$. What would u equal if we had let a be the side opposite θ?

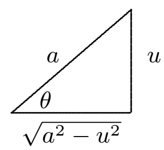


$u=a\sec\theta$

$du=a\sec\theta\tan\theta\,d\theta$

$\sqrt{u^2-a^2}=a\tan\theta$

b) In this case, a must correspond to the hypotenuse of the right triangle. (Why?) We have a choice of how to label the legs, one side $\sqrt{a^2-u^2}$ and the other u. With the selection made below, $u=a\sin\theta$, which is simpler than the other choice.

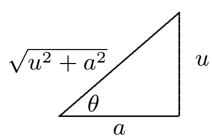


$u=a\sin\theta$

$du=a\cos\theta\,d\theta$

$\sqrt{a^2-u^2}=a\cos\theta$

c) Why must $\sqrt{a^2+u^2}$ must correspond to the hypotenuse of the right triangle? Why choose the u and a sides as follows rather than reverse their positions?



$u=a\tan\theta$

$du=a\sec^2\theta\,d\theta$

$\sqrt{u^2+a^2}=a\sec\theta$

3. All answers are "$+c$"

a) $\sqrt{x^2-1}-\arctan(\sqrt{x^2-1})$ **b)** $\dfrac{x}{36\sqrt{36+x^2}}$

c) $\dfrac{\sqrt{x^2-4}}{4x}$ **d)** $-\frac{1}{3}(4-x^2)^{3/2}$

e) $\ln\left|\dfrac{2+x}{2-x}\right|$ **f)** $-4\sqrt{4-x^2}+\frac{1}{3}(4-x^2)^{3/2}$ or $-\frac{1}{3}x^2\sqrt{4-x^2}-\frac{8}{3}\sqrt{4-x^2}$

4. All "$+c$".

a) $2\ln|x-1|-\ln|x+3|$ **b)** $-\frac{2}{3}\ln|2-x|-\frac{4}{3}\ln|4+x|$

c) $\ln|x|-3\ln|x+2|+2\ln|x-1|$ **d)** $\frac{1}{2}\ln|x^2+8x+2|$ **e)** $\frac{3}{4}\ln|x-2|+\frac{1}{4}\ln|x+2|$

5. a) $\frac{\pi}{12}-\frac{\sqrt{3}}{8}$
b) $\frac{1-\ln 2}{2}$

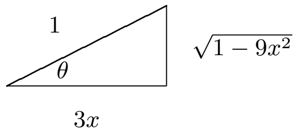
6. πab

7. **a)** $\frac{1}{2}[\sec x\tan x+\ln|\sec x+\tan x|]+c$ **b)** $\frac{7}{2}\sqrt{3}+\frac{1}{8}\ln|7+4\sqrt{3}|+c$

8. **a)** Area $=\frac{1}{5}(\ln 4+\ln 3-\ln 2)=\frac{1}{5}\ln 2+\frac{1}{5}\ln 3$

9. a) u-sub: $u=4-x$. (b) Easiest: u-sub: $u=4-x^2$. Harder: Triangle with $x=2\sin\theta$. (c) Easiest: Partial fractions: $\frac{A}{2-x}+\frac{B}{2+x}$. Harder: Triangle with $x=2\sin\theta$. (d) Easiest: $\arcsin\frac{x}{2}$. Silly: Triangle with $x=2\sin\theta$. (e) Triangle with $x=2\tan\theta$. (f) u-sub: $u=4-x^2$. Harder: Partial fractions: $\frac{A}{2-x}+\frac{B}{2+x}$. Harder: Triangle with $x=2\sin\theta$. (g) Triangle with $x=2\sin\theta$. (h) Partial fractions: $\frac{A}{x+1}+\frac{B}{x+3}$. Harder: Triangle with $x=2\sin\theta$.

Math 131 Lab 09: Answers

1. 

$3x = \cos\theta \Rightarrow x = \frac{1}{3}\cos\theta$

$dx = -\frac{1}{3}\sin\theta\, d\theta$

$\sqrt{1-9x^2} = \sin\theta$ and $9x^2 = \cos^2\theta$

$$\int \frac{1}{9x^2\sqrt{1-9x^2}}\,dx = \int \frac{-\frac{1}{3}\sin\theta}{\cos^2\theta\sin\theta}\,d\theta = -\frac{1}{3}\int \sec^2\theta\,d\theta = -\tfrac{1}{3}\tan\theta + c = -\frac{\sqrt{1-9x^2}}{9x} + c$$

3. Where appropriate, use the triangles in #2 to help with the setup. All "$+c$."

a) $= \displaystyle\int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int \tan^2\theta\,d\theta = \int(\sec^2\theta - 1)\,d\theta = \tan\theta - \theta = \sqrt{x^2-1} - \arctan(\sqrt{x^2-1})$

b) $= \displaystyle\int \frac{1}{6^3\sec^3\theta}6\sec^2\theta\,d\theta = \frac{1}{36}\int\cos\theta\,d\theta = \frac{1}{36}\sin\theta = \frac{x}{36\sqrt{36+x^2}}$

c) $= \displaystyle\int \frac{1}{4\sec^2\theta 2\tan\theta}2\sec\theta\tan\theta\,d\theta = \int \tfrac{1}{4}\cos\theta\,d\theta = \tfrac{1}{4}\sin\theta = \tfrac{\sqrt{x^2-1}}{4x}$

d) Substitution is easiest: $u = 4-x^2$ and $du = -2x\,dx$. So $\displaystyle -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{3}u^{3/2} = -\tfrac{1}{3}(4-x^2)^{3/2}$.

e) Partial fractions: $\displaystyle\frac{4}{4-x^2} = \frac{1}{2-x} + \frac{1}{2+x}$. So $\displaystyle\int \frac{1}{2-x} + \frac{1}{2+x}\,dx = -\ln|2-x| + \ln|2+x| = \frac{\ln|2+x|}{\ln|2-x|}$.

f) $= \displaystyle\int \frac{8\sin^3\theta}{2\cos\theta}2\cos\theta\,d\theta = \int 8\sin^3\theta\,d\theta = 8\int(1-\cos^2\theta)\sin\theta\,d\theta = 8\int\sin\theta - \cos^2\theta\sin\theta\,d\theta = -8\cos\theta + \frac{8}{3}\cos^3\theta =$ $\displaystyle -4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2}$. Or using reduction $= \int 8\sin^3\theta\,d\theta = 8[-\frac{1}{3}\sin^2\theta\cos\theta + \frac{2}{3}\int\sin\theta d\theta] = 8[-\frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta] = -\frac{1}{3}x^2\sqrt{4-x^2} - \frac{8}{3}\sqrt{4-x^2}$.

4. a) $\displaystyle\frac{x+7}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{2}{x-1} - \frac{1}{x+3} \Rightarrow \int \frac{2}{x-1} - \frac{1}{x+3}\,dx = 2\ln|x-1| - \ln|x+3| + c.$

b) $\displaystyle\frac{2x}{(2-x))(4+x)} = \frac{A}{2-x} + \frac{B}{4+x} = \frac{2/3}{2-x} - \frac{4/3}{x+4} \Rightarrow \int \frac{2/3}{2-x} - \frac{4/3}{4+x}\,dx = -\tfrac{2}{3}\ln|2-x| - \tfrac{4}{3}\ln|4+x| + c$

c) $\displaystyle\frac{8x-2}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{1}{x} + \frac{2}{x-1} - \frac{3}{x+2} \Rightarrow \int \frac{1}{x} + \frac{2}{x-1} - \frac{3}{x+2}\,dx$

$$= \ln|x| + 2\ln|x-1| - 3\ln|x+2| + c.$$

d) Substitution: $u = x^2+8x+2$, $du = 2x+8\,dx$. So $= \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+8x+2| + c$.

e) $\displaystyle\frac{x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x-2} = \frac{3/4}{x-2} + \frac{1/4}{x+2} \Rightarrow \int \frac{3/4}{x-2} + \frac{1/4}{x+2}\,dx = \tfrac{3}{4}\ln|x-2| + \tfrac{1}{4}\ln|x+2| + c.$

5. Where appropriate, use the triangles in #1 to help with the setup.

a) Triangle. $x = \sin\theta$ so $x = 0 \Rightarrow \theta = 0$, and $x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.

$$\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}2\cos\theta\,d\theta = \int\sin^2\theta\,d\theta = \int \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\,d\theta = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta)\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

b) u-substitution. $u = t+4$, $du = dt$ $t^2 = (u-4)^2$. $x = 0 \Rightarrow u = 4$, and $x = 5 \Rightarrow u = 9$.

$$\int_4^9 \frac{u^2-8u+16}{u^{1/2}}\,du = \int_4^9 u^{3/2} - 8u^{1/2} + 16u^{-1/2}\,du = \tfrac{2}{5}u^{5/2} - \tfrac{16}{3}u^{3/2} + 32u^{1/2}\Big|_4^9 = [\tfrac{486}{5} - 144 + 96] - [\tfrac{64}{5} - \tfrac{128}{3} + 64] = \tfrac{226}{15}$$

c) $x = \tan\theta$, $dx = \sec^2\theta$, $\sqrt{1+x^2} = \sec\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = 1 \Rightarrow \theta = \frac{\pi}{4}$. So $\displaystyle\int_0^{\pi/4} \frac{\tan^3\theta}{\sec^2\theta}\sec^2\theta\,d\theta =$

$$\int_0^{\pi/4} \tan^3\theta\,d\theta = \int \tan\theta(\sec^2\theta - 1)\,d\theta = \int \tan\theta\sec^2\theta - \tan\theta\,d\theta = \tfrac{1}{2}\tan^2\theta - \ln|\sec\theta|\Big|_0^{\pi/4}$$

$$= [\tfrac{1}{2}(1) - \ln\sqrt{2}] - (0-0) = \tfrac{1}{2} - \ln 2^{1/2} = \tfrac{1-\ln 2}{2}.$$

d) Parts:

$u = \arctan x$	$dv = 3x^2\,dx$	$= x^3 \arctan x\Big\vert_0^1 - \int_0^1 \frac{x^3}{1+x^2}\,dx$
$du = \frac{1}{1+x^2}\,dx$	$v = x^3$	using part (c) $= (\frac{\pi}{4} - 0) - \frac{1-\ln 2}{2} = \frac{\pi}{4} - \frac{1-\ln 2}{2}$

6. $y = b\sqrt{1 - \frac{x^2}{a^2}} = b\sqrt{\frac{a^2-x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}$. Use a triangle substitution. $x = a\sin\theta$, so $x = -a = a\sin\theta \Rightarrow \theta = -\pi/2$ and $x = a = a\sin\theta \Rightarrow \theta = \pi/2$

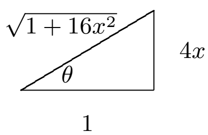
$$A = \int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx = \int_{-\pi/2}^{\pi/2} \frac{b}{a} 2(a^2\cos^2\theta)\,d\theta = \int_{-\pi/2}^{\pi/2} ab[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta)\,d\theta = ab[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta)]\Big|_{-\pi/2}^{\pi/2}$$

$$= ab[\tfrac{\pi}{4} - (-\tfrac{\pi}{4})] = \tfrac{1}{2}\pi ab$$

So the area of the ellipse is πab.

7. a) $\displaystyle\int \sec^3 x\,dx = \tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\int \sec^x x\,dx = \tfrac{1}{2}[\sec\theta\tan x + \ln|\sec x + \tan x|] + c$

b) Use triangles. $f'(x) = 4x$ so $AL = \int_0^{\sqrt{3}} \sqrt{1+16x^2}\,dx = \int \frac{1}{4}\sec^3\theta\,d\theta$. Using part (a)



$$AL = \tfrac{1}{8}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|] = \tfrac{1}{8}[4x\sqrt{1+16x^2}) + \ln|\sqrt{1+16x^2} + 4x|]\Big|_0^{\sqrt{3}}$$

$$= \tfrac{1}{8}[4\sqrt{3}\cdot 7 + \ln|7+4\sqrt{3}| - 0] = \tfrac{7}{2}\sqrt{3} + \tfrac{1}{8}\ln|7+4\sqrt{3}|.$$

8. a) Partial fractions: $A = \displaystyle\int_0^2 \frac{1}{6+x-x^2}\,dx = \int_0^2 \frac{1/5}{3-x} + \frac{1/5}{2+x}\,dx = \tfrac{1}{5}[-\ln|3-x| + \ln|2+x|\Big|_0^2 = \tfrac{1}{5}[\ln 4 + \ln 3 - \ln 2]$

b) Partial fractions: $V = 2\pi\displaystyle\int_0^2 \frac{x}{6+x-x^2}\,dx = 2\pi\int_0^2 \frac{3/5}{3-x} - \frac{2/5}{2+x}\,dx = \tfrac{2\pi}{5}[-3\ln|3-x| - 2\ln|2+x|\Big|_0^2 = \tfrac{2\pi}{5}[-2\ln 4 + 3\ln 3 + 2\ln 2] = \tfrac{6\pi}{5}\ln 3 - \tfrac{2\pi}{5}\ln 2$. (Use $\ln 4 = 2\ln 2$.)

9. a) u-sub: $u = 4 - x$.

b) Easiest: u-sub: $u = 4 - x^2$. Harder: Triangle with $x = 2\sin\theta$.

c) Easiest: Partial fractions: $\frac{A}{2-x} + \frac{B}{2+x}$. Harder: Triangle with $x = 2\sin\theta$.

d) Easiest: $\arcsin\frac{x}{2}$. Silly: Triangle with $x = 2\sin\theta$.

e) Triangle with $x = 2\tan\theta$.

f) u-sub: $u = 4 - x^2$. Harder: Partial fractions: $\frac{A}{2-x} + \frac{B}{2+x}$. Harder: Triangle with $x = 2\sin\theta$.

g) Triangle with $x = 2\sin\theta$.

h) Partial fractions: $\frac{A}{x+1} + \frac{B}{x+3}$. Harder: Triangle with $x = 2\sin\theta$.