Math 131 PracTest 2

0. Let S be the region enclosed by the y axis, $y = x^2 + 4$, and $y = 2x^2$ in the first quadrant only.

a) Sketch the region. Find the area of S. (Ans: 16/3)

b) Rotate S about the x-axis and find the resulting volume. (Ans: $512\pi/15$)

c) Rotate S about the y-axis and find the resulting volume. (Ans: 8π)

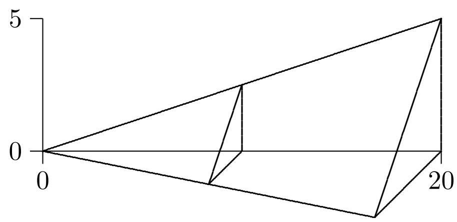
1. Let R be the region in the first quadrant enclosed by enclosed by the x-axis, $y = \sqrt{x}$ and $y = x - 2$. Sketch the region.

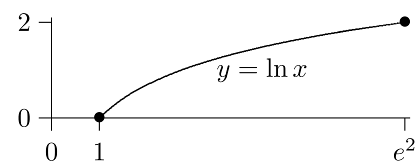
a) Find the area of R. (Ans: 10/3)

b) Rotate R about the x-axis and find the volume. (Ans: $16\pi/3$)

c) Rotate R about the y-axis and find the volume. Try both methods. (Ans: $184\pi/15$)

2. A wooden doorstop with right triangular cross-sections is 20 cm long and 5 cm high at its tall end and 4 cm wide. Find its volume. Hint: Find the equation of the line that forms the top edges and use similar triangles to find the cross-sectional area. (Ans: $\frac{200}{3}$ cm^3.)





3. a) Let R be the region between $y = \ln x$ and the x axis on the interval $[1, e^2]$. Rotate R around the x axis and find the resulting volume. What integration technique should you use? (Answer: $\pi(2e^2 - 1)$.)

b) Rotate R about the y axis and find the volume using the shell method. (Answer: $\frac{\pi}{2}(1 + 3e^4)$.)

c) Rotate R about the y axis and find the volume using the disk method. (Answer: $\frac{\pi}{2}(1 + 3e^4)$.)

4. Let R be the region between $y = \sin \pi x$ and the x axis on the interval $[0, 1]$. Rotate R about the x-axis and find the resulting volume. Hint: Use an identity. (Answer: $\pi/2$.)

5. Find the average value of $f(x) = \sin^3 x$ on the interval $[0, \pi]$. [Ans: $4/3\pi$.]

6. Find the volume of the solid region generated when the area in the first quadrant enclosed by $y = \cos x$, $y = 0$, $x = 0$ and $x = \pi/2$ is revolved around the y-axis. Use shells. [Ans: $\pi^2 - 2\pi$.]

7. a) Find $\int_0^{\pi/4} x \tan^2 x \, dx$. [Ans: $\frac{\pi}{4} - \ln\sqrt{2} - \frac{\pi^2}{32}$.]

b) Let R be the region between $y = \tan^2 x$, $x = \pi/4$, and the x-axis in the first quadrant. Rotate R about the y axis and find the volume using the shell method. Re-use part (a). [Ans: $\frac{\pi^2}{2} - 2\pi\ln\sqrt{2} - \frac{\pi^3}{16}$.]

c) Let R be the region enclosed by $y = \tan^{-1} x$, the y-axis, and $y = \pi/4$ in the first quadrant. Rotate R about the y-axis to form a tank. If it is full of a liquid whose density is 64 lbs/cu. ft., how much work is lost if it leaks out the bottom and drops to ground level? Hint: Use part (a)! [Ans: $-16\pi^2 + 64\pi\ln\sqrt{2} + 2\pi^3$.]

8. Find the length of $f(x) = \frac{4}{3}x^{3/2} + 1$ on the interval $[0, 2]$. (Answer: 13/3)

9. Let $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ on $[2, 3]$. Find the arc length. (Answer: 13/4.)

10. Find the volume of the solid region generated when the area in the first quadrant enclosed by $y = \sqrt{\frac{x}{9+x^2}}$, $y = 0$, and $x = 2$ is revolved around the x-axis. [Ans: $\frac{\pi}{2}[\ln\frac{13}{9}]$]

11. Find the volume of the solid region generated when the area in the first quadrant enclosed by $y = \cos x$, $y = 0$, $x = 0$ and $x = \pi/2$ is revolved around the y-axis. Use shells. [Ans: $\pi^2 - 2\pi$.]

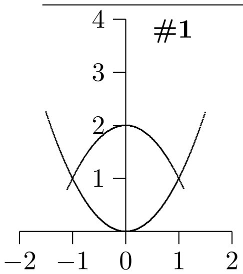
12. a) A small farm elevated water tank is in the shape obtained from rotating the region in the first quadrant enclosed by the curves $y = 10 - \frac{1}{2}x^2$, $y = 8$, and the y-axis about the y-axis. Find the work "lost" if the water (62.5 lbs/ft^3) leaks onto the ground from a hole in the bottom of the tank. (Answer: $-6500\pi/3$ ft-lbs.)

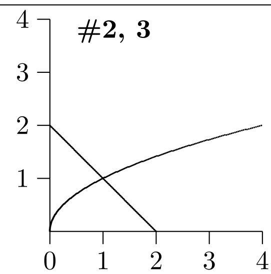
b) Find the work "lost" if the water leaks onto the ground from a hole in the side of the tank at height 9 feet. (Answer: $-1750\pi/3$ ft-lbs.)

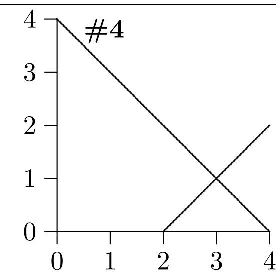
c) Set up integral for the work to empty a tank containing just one foot of water over the top edge.

13. Find the arc length of the parabola $f(x) = \frac{x^2}{2}$ on the interval $[0, 1]$. You will have to use a trig substitution. Make sure that you switch the limits of the integration. You should eventually need to use a reduction formula.

a) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$, the x-axis, and the vertical lines $x = 5$ and $x = 6$.







1. Rotation about the x-axis. Let R be the *entire* region enclosed by $y = x^2$ and $y = 2 - x^2$ in the upper half-plane. Rotate R about the x-axis. The resulting volume is given by:

a) $\pi \int_{-1}^{1} (x^2)^2\,dx - \pi \int_{-1}^{1} (2-x^2)^2\,dx$

b) $\pi \int_{0}^{1} (2-x^2)^2\,dx - \pi \int_{0}^{1} (x^2)^2\,dx$

c) $\pi \int_{-1}^{1} (2-x^2)^2\,dx + \pi \int_{-1}^{1} (x^2)^2\,dx$

d) $\pi \int_{-1}^{1} (2-x^2)^2\,dx - \pi \int_{-1}^{1} (x^2)^2\,dx$

e) $\pi \int_{0}^{1} (\sqrt{y})^2\,dy + \pi \int_{1}^{2} \left(\sqrt{2-y}\right)^2\,dy$

f) None of these

2. Rotation about the x-axis. Let S be the region enclosed by the x-axis, $y = \sqrt{x}$, and $y = 2 - x$. The volume generated by revolving S about the x-axis is:

a) $\pi \int_{0}^{1} \left(\sqrt{x}\right)^2\,dx - \pi \int_{1}^{2} (2-x)^2\,dx$

b) $2\pi \int_{0}^{1} x(2-x)\,dx - 2\pi \int_{0}^{1} x\sqrt{x}\,dx$

c) $\pi \int_{0}^{1} \left(\sqrt{x}\right)^2\,dx + \pi \int_{1}^{2} (2-x)^2\,dx$

d) $\pi \int_{0}^{1} (2-y)^2 - (y^2)^2\,dy$

e) $\pi \int_{0}^{2} \left(\sqrt{x}\right)^2\,dx - \pi \int_{1}^{2} (2-x)^2\,dx$

f) None of these

3. Rotation about the y-axis. Let T be the region enclosed by the y-axis, $y = \sqrt{x}$, and $y = 2 - x$ (a different region than in Problem 2). The volume generated by revolving T about the y-axis is:

a) $\pi \int_{0}^{2} (y^2)^2\,dy - \pi \int_{0}^{2} (2-y)^2\,dy$

b) $2\pi \int_{0}^{1} x(2-x)^2\,dx - 2\pi \int_{0}^{1} x\left(\sqrt{x}\right)^2\,dx$

c) $\pi \int_{0}^{1} \left(\sqrt{x}\right)^2\,dx + \pi \int_{1}^{2} (2-x)^2\,dx$

d) $\pi \int_{0}^{1} (2-y)^2\,dy + \pi \int_{1}^{2} (y^2)^2\,dy$

e) $2\pi \int_{0}^{1} x(2-x)\,dx - 2\pi \int_{0}^{1} x(\sqrt{x})\,dx$

f) None of these

4. Rotation about the y-axis. Let U be the region enclosed by the y-axis, the x-axis, $y = x - 2$, and $y = 4 - x$. The volume generated by revolving U about the y-axis is:

a) $\pi \int_{0}^{1} (y-2)^2 + \pi \int_{1}^{4} (4-y)^2\,dy$

b) $2\pi \int_{0}^{3} x(4-x)^2\,dx - 2\pi \int_{2}^{3} x(x-2)^2\,dx$

c) $2\pi \int_{0}^{4} x(4-x)\,dx - 2\pi \int_{2}^{3} x(x-2)\,dx$

d) a and c

e) a and b

f) None of these

Practest 2, Part 2.

1. Integral Mix Up: Gotta game plan?

a) $\int (4x^3+1)\ln x\,dx$ **b)** $\int xe^{x+1}\,dx$ **c)** $\int e^{2x}\cos x\,dx$

d) $\int x\sec^2 x\,dx$ **e)** $\int \cos^3(4x)\,dx$

f) $\int \ln(2x^3)\,dx$ **g)** $\int \sin^2(5x)\,dx$ **h)** $\int x^2\ln x^2\,dx$

i) $\int \sin^3 x\cos^2 x\,dx$ **j)** $\int \arctan 2x\,dx$ **k)** $\int 2\sec^3 x\tan x\,dx$

l) $\int x\sin^2 x\,dx$

2. Find the area enclosed by the curves $y=x^3+x$ and $y=3x^2-x$.

3. Determine $\int 2x\arcsin x\,dx$. You will need to use several different methods.

4. Draw the right triangle associated with each of these square roots and label the sides. For each, solve for u, du, and the given square root in terms of an angle θ.

a) $\sqrt{u^2-a^2}$ **b)** $\sqrt{a^2-u^2}$ **c)** $\sqrt{u^2+a^2}$

5. Determine these integrals. *Caution:* A variety of techniques are required. Where necessary, make use of the triangles that you just drew.

a) $\int \frac{\sqrt{x^2-1}}{x}\,dx$ **b)** $\int \frac{1}{(36+x^2)^{3/2}}\,dx$ **c)** $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$

d) $\int x\sqrt{4-x^2}\,dx$ **e)** $\int \frac{4}{4-x^2}\,dx$ **f)** $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ **g)** $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

6. Determine $\int \frac{1}{9x^2\sqrt{1-9x^2}}\,dx$.

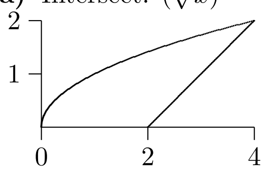
Math 131 Practest 2: Selected Answers

1. a) $A = \int_0^2 (x^2+4) - 2x^2\,dx = \int_0^2 4 - x^2\,dx = 4x - \frac{1}{3}x^3\Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$.

b) $V = \pi\int_0^2 (x^2+4)^2 - (2x^2)^2\,dx = \pi\int_0^2 (x^4+8x^2+16) - 4x^4\,dx = \pi\int_0^2 8x^2 + 16 - 3x^4\,dx = \pi[\frac{8}{3}x^3 + 16x - \frac{3}{5}x^5]\Big|_0^2 = \pi[(\frac{64}{3} + 32 - \frac{96}{5}) - 0] = 512\pi/15$

c) $V = 2\pi\int_0^2 x(x^2+4) - x(2x^2)\,dx = 2\pi\int_0^2 (x^3+4x) - 2x^3\,dx = 2\pi\int_0^2 4x - x^3\,dx = 2\pi[2x^2 - \frac{1}{4}x^4]\Big|_0^2 = 2\pi[(8-4] = 8\pi$

2. a) Intersect: $(\sqrt{x})^2 = (x-2)^2 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow x = 4$ (not $x = 1$). Note $x = y^2$ and $x = y + 2$



$A = \int_0^4 x^{1/2}\,dx - \int_2^4 x - 2\,dx$

$= \frac{2}{3}x^{3/2}\Big|_0^4 - (\frac{1}{2}x^2 - 2x)\Big|_2^4 = [\frac{16}{3} - 0] - [0 - (-2)] = 10/3$

b) $V = \pi\int_0^4 (x^{1/2})^2\,dx - \pi\int_2^4 (x-2)^2\,dx = \pi[\frac{1}{2}x^2]\big|_0^4 - \pi[\frac{1}{3}(x-2)^3]\big|_2^4 = \pi[(8-0) - (\frac{8}{3} - 0)] = 16\pi/3$

c) $V = \pi\int_0^2 (y+2)^2 - (y^2)^2\,dy = \pi[\frac{1}{3}(y+2)^3 - \frac{1}{5}y^5]\big|_0^2 = \pi[(\frac{64}{3} - \frac{32}{5} - (\frac{8}{3} - 0)] = 184\pi/15$

3. The line for the height is $y = \frac{5}{20}x = \frac{1}{4}x$. The line for the base width is $y = \frac{4}{20}x = \frac{1}{5}x$. So the cross-sectional area is $A(x) = \frac{1}{2}bh = \frac{1}{2}(\frac{1}{5}x)(\frac{1}{4}x) = \frac{1}{40}x^2$. So

$$V = \int_0^{20} \frac{1}{40}x^2\,dx = \frac{1}{120}x^3\Big|_0^{20} = \frac{200}{3}\text{ cm}^3.$$

4. a) Parts twice:

$u = (\ln x)^2$ $du = \frac{2\ln x}{x}\,dx$	$dv = dx$ $v = x$	$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$
$u = 2\ln x$ $du = \frac{2}{x}\,dx$	$dv = dx$ $v = x$	$\int (\ln x)^2 x\,dx = x(\ln x)^2 - \left[2x\ln x - \int 2\,dx\right] = x(\ln x)^2 - 2x\ln x + 2x + c$

So $\pi\int_1^{e^2} (\ln x)^2\,dx = \pi[x(\ln x)^2 - 2x\ln x + 2x]\Big|_1^{e^2} = \pi[(4e^2 - 4e^2 + 2e^2) - (0 - 0 + 1) = \pi(2e^2 - 1)$.

b) Parts:

$u = \ln x$ $du = \frac{1}{x}\,dx$	$dv = 2\pi x\,dx$ $v = \pi x^2$	$\int_1^{e^2} 2\pi x\ln x\,dx = \pi x^2\ln x\Big|_1^{e^2} - \int_1^{e^2} \pi x\,dx$ $= \pi x^2\ln x - \frac{\pi}{2}x^2\Big|_1^{e^2} = (2\pi e^4 - \frac{\pi}{2}e^4) - (0 - \frac{\pi}{2}) = \frac{3\pi}{2}e^4 + \frac{\pi}{2}$

c) $V = \pi\int_0^2 (e^2)^2 - (e^y)^2\,dy = \pi\int_0^2 e^4 - e^{2y}\,dy = \pi[e^4 y - \frac{1}{2}e^{2y}\big|_0^2 = \pi[(2e^4 - \frac{1}{2}e^4) - (0 - \frac{1}{2})] = \frac{3\pi}{2}e^4 + \frac{\pi}{2}$.

5. $V = \pi\int_0^1 \sin^2(\pi x)\,dx = \pi\int_0^1 \frac{1}{2} - \frac{1}{2}\cos(2\pi x)\,dx = \pi[\frac{1}{2}x - \frac{1}{4\pi}\sin(2\pi x)]\big|_0^1 = \pi[(\frac{1}{2} - 0) - 0] = \frac{\pi}{2}$

6. Ave Val $= \frac{1}{\pi - 0}\int_0^\pi \sin^3 x\,dx = \frac{1}{\pi}\int_0^\pi (1 - \cos^2 x)\sin x\,dx = \frac{1}{\pi}\int_0^\pi \sin x - \cos^2 x\sin x\,dx = \frac{1}{\pi}[-\cos x + \frac{1}{3}\cos^3 x]\big|_0^\pi = \frac{1}{\pi}[(1 - \frac{1}{3}) - (-1 + \frac{1}{3})] = \frac{4}{3\pi}$.

7. $V = \int_0^{\pi/2} 2\pi x\cos x\,dx = 2\pi[x\sin x - \int \sin x\,dx]\big|_0^{\pi/2} = 2\pi[x\sin x + \cos x\,dx]\big|_0^{\pi/2} = \pi^2 - 2\pi$.

8. a) Trig id and then parts: $\int_0^{\pi/4} x\tan^2 x\,dx = \int_0^{\pi/4} x(\sec^2 x - 1)\,dx = \int_0^{\pi/4} x\sec^2 x - x\,dx$. To do $\int x\sec^2 x\,dx$:

$u = x$ $du = dx$	$dv = \sec^2 x dx$ $v = \tan x$	$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$ $\int x\sec^2 x\,dx = x\tan x - \ln\lvert\sec x\rvert$

So $\int_0^{\pi/4} x\tan^2 x\,dx = \int_0^{\pi/4} x\sec^2 x - x\,dx = [x\tan x - \ln\lvert\sec x\rvert - \frac{1}{2}x^2]\big|_0^{\pi/4} = \frac{\pi}{4} - \ln\sqrt{2} - \frac{\pi^2}{32}$.

b) $V = \int_0^{\pi/4} 2\pi x\tan^2 x\,dx = 2\pi(\frac{\pi}{4} - \ln\sqrt{2} - \frac{\pi^2}{32}) = \frac{\pi^2}{2} - 2\pi\ln\sqrt{2} - \frac{\pi^3}{16} = \frac{\pi^2}{2} - \pi\ln 2 - \frac{\pi^3}{16}$.

c) Note: $x = \tan y$, so $W = 64\int_0^{\pi/4} (0 - y)\pi\tan^2 y\,dy = -64\pi(\frac{\pi}{4} - \ln\sqrt{2} - \frac{\pi^2}{32}) = -16\pi^2 + 64\pi\ln\sqrt{2} + 2\pi^3$.

9. $AL = \int_0^2 \sqrt{1+4x}\,dx = \frac{1}{4}\cdot\frac{2}{3}(1+4x)^{3/2}\Big|_0^2 = \frac{1}{6}(27-1) = \frac{13}{3}$.

10. $AL = \int_2^3 \sqrt{1+(\frac{1}{2}x^2-\frac{1}{2}x^{-2})^2}\,dx = \int_2^3 \sqrt{(\frac{1}{2}x^2+\frac{1}{2}x^{-2})^2}\,dx = \int_2^3 \frac{1}{2}x^2+\frac{1}{2}x^{-2}\,dx = \frac{1}{6}x^3 - \frac{1}{2}x^{-1}\Big|_2^3 = \frac{13}{4}$.

11. $V = \pi\int_0^2 \left(\sqrt{\frac{x}{9+x^2}}\right)^2 dx = \pi\int_0^2 \frac{x}{9+x^2}\,dx = \frac{\pi}{2}\ln(9+x^2)\Big|_0^2 = \frac{\pi}{2}[\ln 13 - \ln 9]$.

12. $V = \int_0^{\pi/2} 2\pi x\cos x\,dx = 2\pi[x\sin x - \int \sin x\,dx]\Big|_0^{\pi/2} = 2\pi[x\sin x + \cos x\,dx]\Big|_0^{\pi/2} = \pi^2 - 2\pi$.

13. a) Save work! Since the radius of a cross-section of the tank is x, we need to solve for x^2 (not x): but $y = 10 - \frac{1}{2}x^2$, so $\frac{1}{2}x^2 = 10 - y$ or $x^2 = 20 - 2y$.

$$\begin{aligned} W &= 62.5\int_8^{10} \pi(20-2y)(0-y)\,dy = 62.5\pi\int_8^{10} 2y^2 - 20y\,dy = 62.5\pi[2y^3/3 - 10y^2]\Big|_8^{10} \\ &= 62.5\pi[(2000/3-1000)-(1024/3-640)] = -6500\pi/3 \text{ lbf} \end{aligned}$$

b) Only the lower limit changes to 9 since the upper part of the barrel leaks out:

$$\begin{aligned} W = 62.5\int_9^{10} \pi(20-2y)(0-y)\,dy &= -62.5\pi[2y^3/3-10y^2]\Big|_9^{10} \\ &= -62.5\pi[(2000/3-1000)-(486-810)] = -1750\pi/3 \text{ lbf} \end{aligned}$$

c) Note the limits and the height 'moved to': $W = 62.5\int_8^9 \pi(20-2y)(10-y)\,dy$

14. $f'(x) = x$, so $AL = \int_0^1 \sqrt{1+x^2}\,dx$. Use triangles. $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{1+x^2} = \sec\theta$. $x = 0 = \tan\theta \Rightarrow \theta = 0$. $x = 1 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$. So

$$AL = \int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \sec\theta\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta.$$

Use a Reduction Formula:

$$\int \sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int\sec\theta\,d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\ln|\sec\theta+\tan\theta| + c$$

So

$$\int_0^{\pi/4}\sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2}\Bigg|_0^{\pi/4} = \frac{\sqrt{2}+\ln(1+\sqrt{2})}{2}.$$

15. a) Triangles. $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int\frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|$$

So

$$\int_5^6 \frac{1}{\sqrt{x^2-9}}\,dx = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|\Bigg|_5^6 = \ln\left|\frac{6}{3}+\frac{\sqrt{27}}{3}\right| - \ln\left|\frac{5}{3}+\frac{4}{3}\right| = \ln\left(2+\sqrt{3}\right) - \ln 3$$

Quiz Answers

1) (d) **2)** (c) **3)** (e) **4)** (f)

1. a)

$u = \ln x$	$dv = 4x^3+1\,dx$	$\int(4x^3+1)\ln x\,dx = (x^4+x)\ln x - \int\frac{x^4+x}{x}\,dx$
$du = \frac{1}{x}\,dx$	$v = x^4+x$	$= (x^4+x)\ln x - \int x^3+1\,dx = (x^4+x)\ln x - \frac{1}{4}x^4 - x + c\,dx$

b)

$u = x$ $\quad du = dx$	$dv = e^{x+1}\,dx$ $\quad v = e^{x+1}$	$\int xe^{x+1}\,dx = xe^{x+1} - \int e^{x+1}\,dx = xe^{x+1} - e^{x+1} + c = e^{x+1}(x-1) + c$

c) Parts twice, "circle around". Note signs and constants:

$u = e^{2x}$ $\quad du = 2e^{2x}dx$	$dv = \cos x\,dx$ $\quad v = \sin x$	$\int e^{2x}\cos x\,dx = e^{2x}\sin x - \int 2e^{2x}\sin x\,dx$
$u = 2e^{2x}$ $\quad du = 4e^{2x}dx$	$dv = \sin x\,dx$ $\quad v = -\cos x$	$\int e^{2x}\sin x\,dx = e^{2x}\sin x - \left[-2e^{2x}\cos x + 4\int e^{2x}\cos x\,dx\right]$ So, $5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x + c$ Thus, $\int e^{2x}\cos x\,dx = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$

d)

$u = x$ $\quad du = dx$	$dv = \sec^2 x\,dx$ $\quad v = \tan x$	$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln\lvert\cos x\rvert + c$

e) Reduction: $u = 4x$ so $\frac{1}{4}du = dx$. $\displaystyle\int \cos^3(4x)\,dx = \frac{1}{4}\int \cos^3 u\,du$

$$= \frac{1}{4}\left[\frac{\cos^2 u\sin u}{3} + \frac{2}{3}\int \cos u\,du\right] = \frac{1}{4}\left[\frac{\cos^2 u\sin u}{3} + \frac{2}{3}\sin u\right] = \frac{1}{12}\cos^2(4x)\sin(4x) + \frac{1}{6}\sin(4x) + c$$

f)

$u = \ln(2x^3)$ $\quad du = \frac{6x^2}{2x^3}\,dx = \frac{3}{x}\,dx$	$dv = dx$ $\quad v = x$	$\int \ln(2x^3)\,dx = x\ln(2x^3) - \int \frac{3x}{x}\,dx$ $= x\ln(2x^3) - \int 3\,dx = x\ln(2x^3) - 3x + c$

g) $\int \sin^2(5x)\,dx = \int \frac{1}{2} - \frac{1}{2}\cos(10x)\,dx = \frac{1}{2}x - \frac{1}{20}\sin(10x) + c.$

h)

$u = \ln(x^2)$ $\quad du = \frac{2x}{x^2}\,dx = \frac{2}{x}\,dx$	$dv = x^2\,dx$ $\quad v = \frac{1}{3}x^3$	$\int x^2\ln x^2\,dx = \frac{1}{3}x^3\ln(x^2) - \int \frac{2x^3}{3x}\,dx$ $= \frac{1}{3}x^3\ln(x^2) - \int \frac{2}{3}x^2\,dx = \frac{1}{3}x^3\ln(x^2) - \frac{2}{9}x^3 + c$

i) $\int \sin^3 6x\cos^2 6x\,dx = \int \sin^2 6x\cos^2 6x\sin 6x\,dx = \int (1-\cos^2 6x)\cos^2 6x\sin 6x\,dx = -\frac{1}{6}\int u^2 - u^4\,du = -\frac{1}{6}\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + c = \frac{\cos^5 6x}{30} - \frac{\cos^3 6x}{18} + c$

j) Parts then substitution:

$u = \arctan(2x)$ $\quad du = \frac{2}{1+4x^2}\,dx$	$dv = dx$ $\quad v = x$	$\int \arctan(2x)\,dx = x\arctan(2x) - \int \frac{2x}{1+4x^2}\,dx$
$u = 4x^2$	$du = 4x\,dx$ $\quad \frac{1}{8}\,du = 2x\,dx$	So, $\int \frac{2x}{1+4x^2}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln\lvert u\rvert = \frac{1}{4}\ln(1+4x^2)$, so $\int \arctan(2x)\,dx = x\arctan(2x) - \frac{1}{4}\ln(1+4x^2) + c$

k) $\int 2\sec^3 x\tan x\,dx = 2\int \sec^2 x(\sec x\tan x)\,dx = \frac{2}{3}\sec^3 x\,dx.$

l) Trig ID, then parts: $\int x\sin^2 x\,dx = \int x[\frac{1}{2} - \frac{1}{2}\cos(2x)]\,dx = \int \frac{1}{2}x\,dx - \int \frac{1}{2}x\cos(2x)\,dx.$

$u = \frac{1}{2}x$ $\quad du = \frac{1}{2}\,dx$	$dv = \cos(2x)dx$ $\quad v = \frac{1}{2}\sin(2x)$	$\int \frac{1}{2}x\cos(2x)\,dx = \frac{1}{4}x\sin(2x) - \int \frac{1}{4}\sin(2x)\,dx = \frac{1}{4}x\sin(2x) + \frac{1}{8}\cos(2x)$

Therefore: $\int x\sin^2 x\,dx = \int \frac{1}{2}x\,dx - \int \frac{1}{2}x\cos(2x)\,dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x).$

2. $A_1 = \int_0^1 (x^3+x)-(3x^2-x)\,dx = \int_0^1 x^3-3x^2+2x\,dx = \frac{1}{4}x^4-x^3+x^2\Big|_0^1 = (\frac{1}{4}-1+1)-0 = \frac{1}{4}$. $A_2 = \int_1^2 (3x^2-x)-(x^3+x)\,dx = -\frac{1}{4}x^4 + x^3 - x^2\Big|_1^2 = (-4+8-4) - (-\frac{1}{4}+1-1) = \frac{1}{4}$. $A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

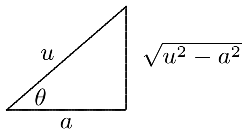
3. First use parts:

$u = \arcsin x$ $\quad du = \frac{1}{\sqrt{1-x^2}}\,dx$	$dv = 2xdx$ $\quad v = x^2$	$\int 2x\arcsin x\,dx = x^2\arcsin x - \int \frac{x^2}{\sqrt{1-x^2}}\,dx$ Now use triangle substitution

$x = \sin\theta$ $\quad \sqrt{1-x^2} = \cos\theta$	$dx = \cos\theta d\theta$ $\quad \theta = \arcsin x$	$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta\cos\theta}{\cos\theta}\,d\theta = \int \sin^2\theta\,d\theta = \int \frac{1}{2} - \frac{1}{2}\cos(2\theta)\,d\theta$ $= \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta) = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + c$

So putting it all together $\displaystyle\int 2x\arcsin x\,dx = x^2\arcsin x - \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + c.$

4. a) In this case, u must correspond to the hypotenuse of the right triangle. (Why?) We have our choice of how to label the legs, one side a and the other $\sqrt{u^2-a^2}$. With the selection below, $u = a\sec\theta$. What would u equal if we had let a be the side opposite θ?

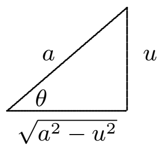


$u = a\sec\theta$

$du = a\sec\theta\tan\theta\,d\theta$

$\sqrt{u^2-a^2} = a\tan\theta$

b) In this case, a must correspond to the hypotenuse of the right triangle. (Why?) We have a choice of how to label the legs, one side $\sqrt{a^2-u^2}$ and the other u. With the selection below, $u = a\sin\theta$, which is simpler than the other choice.

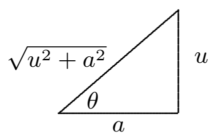


$u = a\sin\theta$

$du = a\cos\theta\,d\theta$

$\sqrt{a^2-u^2} = a\cos\theta$

c) Why must $\sqrt{a^2+u^2}$ must correspond to the hypotenuse of the right triangle? Why choose the u and a sides as follows rather than reverse their positions?



$u = a\tan\theta$

$du = a\sec^2\theta\,d\theta$

$\sqrt{u^2+a^2} = a\sec\theta$

5. Where appropriate, use the triangles in the previous problem to help with the setup.

a) $= \displaystyle\int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int \tan^2\theta\,d\theta = \int(\sec^2\theta-1)\,d\theta = \tan\theta-\theta = \sqrt{x^2-1}-\arctan(\sqrt{x^2-1})+c$

b) $= \displaystyle\int \frac{1}{6^3\sec^3\theta}6\sec^2\theta\,d\theta = \frac{1}{36}\int\cos\theta\,d\theta = \frac{1}{36}\sin\theta = \frac{x}{36\sqrt{36+x^2}}+c$

c) $= \displaystyle\int \frac{1}{4\sec^2\theta 2\tan\theta}2\sec\theta\tan\theta\,d\theta = \int \tfrac{1}{4}\cos\theta\,d\theta = \tfrac{1}{4}\sin\theta = \tfrac{\sqrt{x^2-4}}{4x}+c$

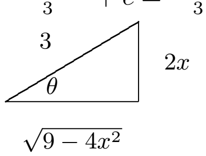
d) Substitution is easiest: $u = 4-x^2$ and $du = -2x\,dx$. So $-\dfrac{1}{2}\displaystyle\int u^{1/2}\,du = -\frac{1}{3}u^{3/2} = -\tfrac{1}{3}(4-x^2)^{3/2}+c.$

e) Triangles: $\displaystyle\int\frac{4}{4-x^2}\,dx = \int\frac{4}{(\sqrt{4-x^2})^2}\,dx = \int\frac{4}{(2\cos\theta)^2}\cdot 2\cos\theta\,d\theta = \int\frac{8\cos\theta}{4\cos^2\theta}\,d\theta = 2\int\sec\theta\,d\theta$

$\displaystyle = 2\int \ln|\sec(\theta)+\tan\theta|\,d\theta = 2\ln|\tfrac{2}{\sqrt{4-x^2}}+\tfrac{x}{\sqrt{4-x^2}}|+c = 2\ln|\tfrac{2+x}{\sqrt{4-x^2}}|+c = \ln|\tfrac{(2+x)^2}{4-x^2}|+c = \ln|\tfrac{2+x}{2-x}|+c$

f) $= \int\frac{8\sin^3\theta}{2\cos\theta}2\cos\theta\,d\theta = \int 8\sin^3\theta\,d\theta = 8\left[-\frac{\sin^2\theta\cos\theta}{3}+\frac{2}{3}\int\sin\theta\,d\theta\right] = 8\left[-\frac{\sin^2\theta\cos\theta}{3}-\frac{2}{3}\cos\theta\right]+c = -\frac{x^2\sqrt{4-x^2}}{3} - \frac{8\sqrt{4-x^2}}{3}+c = -\frac{1}{3}\sqrt{4-x^2}(x^2+8)+c$

g)



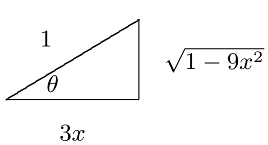
$\frac{2x}{3} = \sin\theta \Rightarrow x = \frac{3}{2}\sin\theta$

$dx = \frac{3}{2}\cos\theta\,d\theta$

$\frac{\sqrt{4-9x^2}}{3} = \cos\theta \Rightarrow \sqrt{4-9x^2} = 3\cos\theta$

$\int\sqrt{9-4x^2}\,dx = \frac{9}{2}\int\cos^2\theta\,d\theta = \frac{9}{2}\int\frac{1}{2}-\frac{1}{2}\cos 2\theta\,d\theta = \frac{9}{2}\left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]+c = \frac{9}{2}\left[\frac{\theta}{2}-\frac{1}{2}\sin\theta\cos\theta\right]+c$
$= \frac{9}{2}\left[\frac{\arcsin(2x/3)}{2}-\frac{1}{2}\frac{2x}{3}\cdot\frac{\sqrt{9-4x^2}}{3}\right]+c = \frac{9}{4}\arcsin(2x/3)-\frac{1}{2}x\sqrt{9-4x^2}+c$

6.



$3x = \cos\theta \Rightarrow x = \frac{1}{3}\cos\theta$

$dx = -\frac{1}{3}\sin\theta\,d\theta$

$\sqrt{1-9x^2} = \sin\theta$ and $9x^2 = \cos^2\theta$

$$\int\frac{1}{9x^2\sqrt{1-9x^2}}\,dx = \int\frac{-\frac{1}{3}\sin\theta}{\cos^2\theta\sin\theta}\,d\theta = -\frac{1}{3}\int\sec^2\theta\,d\theta = -\tfrac{1}{3}\tan\theta+c = -\frac{\sqrt{1-9x^2}}{9x}+c$$