Math 131 Lab 6

There are a half-dozen **types** of volume problems. The type determines the method used, *viz.*, which axis is used for integration (this may be different than the axis of rotation). Sketch the regions and draw a representative slice or radius for each problem. Set up the integrals and simplify the integrands. Only after you have set most of these up, go back and actually calculate the definite integral. **Have someone check your have set-ups!** Some of these are WeBWorK problems.

1. Rotation about the x-axis. Let R be the region in the first quadrant enclosed by $y = \sqrt{x-1}$, $y = x - 7$ and the x-axis. Sketch the region. Rotate R about the x-axis and find the resulting volume. (Ans: $\frac{63}{2}\pi$.)

2. Rotation about the y-axis. Let R be the region enclosed by the x-axis, $y = \sqrt{x}$, and $y = 2 - x$. Rotate R about the y-axis and find the volume. Your choice: disks or shells. (Answer: $32\pi/15$.)

3. Let R be the region enclosed by $y = 2x$ and $y = x^2$.

- **a)** Do later. **Rotation about the x-axis.** Find the volume of the hollowed-out solid generated by revolving R about the x-axis. (Answer: $64\pi/15$)
- **b)** *Do now.* **Rotation about the y-axis (disks).** Find the volume of the solid generated by revolving R about the y-axis by using the disk method and integrating along the y-axis.
- **c)** *Do now.* **Rotation about the y-axis (shells).** This is a WeBWorK problem. Redo the volume of the solid generated by revolving R about the y-axis by using the shell method. Which method was easier?

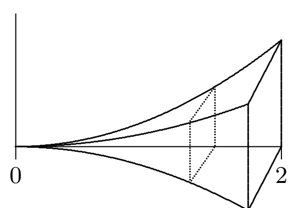
The next two problems are rotations about the y-axis, so either disks or shells are possible. **However**, in each case only one of these methods is easy. That's why it is important to know both.

4. Rotation about the y-axis. This is a WeBWorK problem. Let R be the region in the first quadrant enclosed by $y = e^{3x^2}$, $x = 0$, $x = 1$ and $y = 0$. Rotate this region about the y-axis and find the volume.

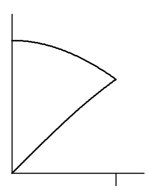
5. a) Rotation about the y-axis. Let R be the region in the first quadrant enclosed by $y = \ln x$, $y = 0$, and $x = e$. Rotate this region about the y-axis and find the resulting volume. (Answer: $\pi(e^2 + 1)/2$.)

- **b) Extra Credit for Later.** Rotate this region about the x-axis and find the resulting volume. Hint: There is a problem on Exam 1 that can help. This is not easy.

6. Non-rotation problem. A crystal prism is 2 cm long (below). Its cross-sections are **squares** with heights are formed by the curve $y = x^2$. Find the volume of the prism. (Answer: 6.4 cu. cm.)



Problem 6



Problem 7

Height (h)	0	40	80	120	160	200
Radius (r)	28	30	26	24	20	18

Problem 10

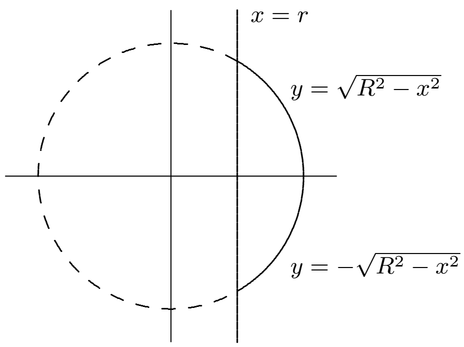
7. Let R be the region enclosed by $y = \cos x$ and $y = \sin x$ on $[0, \pi/4]$ shown above. Rotate R about the $x - axis$ and find the volume. How will you do the antiderivative? (Answer: $\pi/2$.)

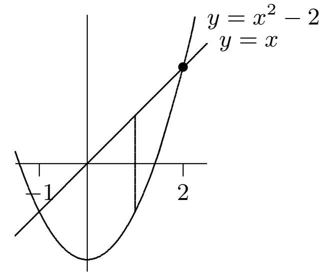
8. Harder rotation about the y-axis. Let R be the region in the *right half-plane* enclosed by $y = 4 - x^2$, $y = x^4 - 4x^2$, and the y-axis. Rotate this region about the y-axis and find the volume. Disks or shells: Only one method is possible. WeBWorK

9. Rotation about the x-axis. Let R be the *entire* region enclosed by $y = x^2$ and $y = 2 - x^2$ in the upper half-plane. Sketch the region. Rotate R about the x-axis and find the resulting volume. (Ans: $\frac{16}{3}\pi$.)

10. A field biologist is doing a survey of a small wooded forest. She is interested in finding the volume of tree trunks from the forest floor to a point 2 meters above the ground. Since she cannot measure the volume directly, she uses a pair of tree calipers to measure the radius of the tree at 40 cm intervals over the range from 0 to 200 centimeters. She brings the data to you (see table above) and asks you to provide a reliable estimate on the volume of the tree trunk in cubic centimeters. How can you do so using Riemann sums? What estimate would be a reasonable maximum estimate? Minimum estimate? Explain your reasoning.

11. Consider the region shown below (left) where $r < R$. It consists of part of a circle or radius R cut off by a line $x = r$. Rotate this region around the y-axis to form a sphere with a hole drilled out of it. Find the resulting volume using the shell method. What integration technique is helpful?





12. Consider the region enclosed by the graphs of $f(x) = x^2 - 2$ and $g(x) = x$ as shown above (right). A solid is formed above this region. The cross-sectional slices perpendicular to the x-axis are semi-circles each of whose diameters the distance between $f(x)$ and $g(x)$. Just set up the integral for the volume of this solid.

13. Extra Fun

a) Let R be the region enclosed by $y = \arctan x$, $y = \pi/4$, and the y-axis. Find the area of R. Hint: Only integration along one axis is possible at this point.

b) Rotate R about the y-axis and find the resulting volume. Use a trig ID for $\tan^2 \theta$.

14. Do After Lab: Extra Credit for Next Class

a) Let $y = \frac{1}{x}$ on $[1, a]$, where $a > 1$. Let R be the region under the curve over this interval. Rotate R about the x-axis. What value of a gives a volume of $\pi/2$?

b) What happens to the volume if $a \to \infty$? Does the volume get infinitely large? Use limits to answer the question.

c) Instead rotate R about the y-axis. What value of a gives a volume of $\pi/2$? This is particularly easy to do by shells!

d) Instead rotate R about the line $y = -1$. If we let $a = e$ what is the resulting volume? Hint: Draw the figure and determine a function describing the radial distance from the axis of rotation.

Additional Practice for Another Time

15. The Great Pyramid of Geneva. The pryamid at the Pyramid Mall in Geneva at the crest of Bean's Hill has a square base with edges that measure 300 meters. Its height is 150 meters. Find its volume. Hint: Review the Pyramid of Cheops problem. The equations are simpler if you turn the pyramid upside down and use cross-sections perpendicular to the y-axis. (Answer: 4,500,000 cu. m.) In order to simplify the mathematics, it will be useful to draw the pyramid upside-down (see below).

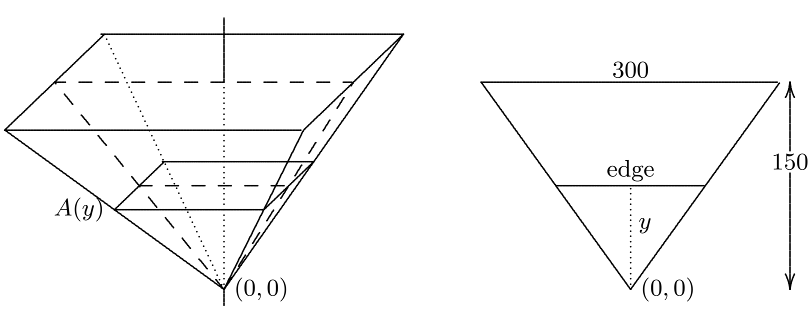


Figure 1: Left: The pyramid at the Pyramid Mall upside down. Cross-sections perpendicular to the y-axis are squares. Right: The relation between the height of the cross-section and its edge length.

16. Let R be the region enclosed by $y = \sqrt{x}$, $y = 2$, and the y-axis.

a) Rotate R about the x-axis and find the resulting volume. (Answer: 8π)

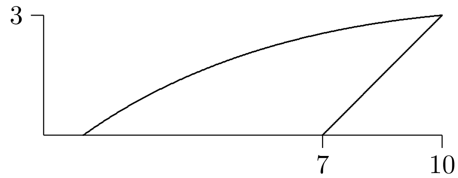
b) Rotate R about the line $y = 2$ and find the resulting volume. (Answer: $8\pi/3$)

c) Rotate R about the line $y = 4$ and find the resulting volume. (Answer: $40\pi/3$)

Math 131 Answers to Lab 6

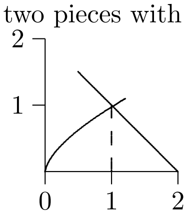
0. $\lim_{n\to\infty} S_n = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$

1. Intersect: $\sqrt{x-1} = x-7 \Rightarrow x-1 = x^2-14x+49 \Rightarrow x^2-15x+50 = (x-10)(x-5) = 0 \Rightarrow x = 10$, not $x = 5$. The line meets the x axis at $x = 7$ and $\sqrt{x-1}$ meets the axis at $x = 1$. Subtract the inside from the outside piece:



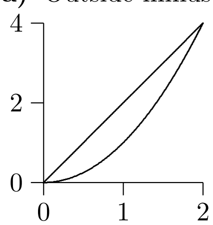
$$\begin{aligned} V &= \pi \int_1^{10} (\sqrt{x-1})^2\,dx - \pi \int_7^{10} (x-7)^2\,dx \\ &= \pi \int_1^{10} x-1\,dx - \pi \int_7^{10} (x-7)^2\,dx \\ &= \pi[\tfrac{x^2}{2} - x]\Big|_1^{10} - \pi[\tfrac{(x-7)^3}{3}]\Big|_7^{10} \\ &= \pi[(50-10) - (\tfrac{1}{2} - 1))] - \pi[9-0] = \tfrac{63}{2}\pi. \end{aligned}$$

2. Intersect: $\sqrt{x} = 2-x \Rightarrow x = 4-4x+x^2 \Rightarrow x^2-5x+4 = (x-1)(x-4) = 0 \Rightarrow x = 1$ not $x = 4$. The integral is in two pieces with shells:



$$\begin{aligned} V &= 2\pi \int_0^1 x\sqrt{x}\,dx + 2\pi \int_1^2 x(2-x)\,dx \\ &= 2\pi[\tfrac{2}{5}x^{5/2}]\Big|_0^1 + 2\pi[x^2 - \tfrac{1}{3}x^3]\Big|_1^2 \\ &= 2\pi[\tfrac{2}{5} - 0] + 2\pi[(4-\tfrac{8}{3}) - (1-\tfrac{1}{3})] = \tfrac{32}{15}\pi. \end{aligned}$$

3. a) Outside minus inside volume.



$$\begin{aligned} V &= \pi \int_0^2 (2x)^2 - (x^2)^2\,dx = \pi \int_0^2 4x^2 - x^4\,dx \\ &= \pi[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5]\Big|_0^2 \\ &= \pi[(\tfrac{32}{3} - \tfrac{32}{5}) - 0] = \tfrac{64}{15}\pi. \end{aligned}$$

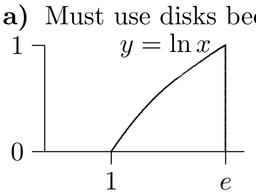
b) Disks: $y = 2x$ and $y = x^2$ so $x = \frac{1}{2}y$ and $x = \sqrt{y}$. Outside minus inside volume.

$$V = \pi \int_0^4 (\sqrt{y})^2 - (\tfrac{1}{2}y)^2\,dy = \pi \int_0^4 y - \tfrac{1}{4}y^2\,dy = \pi[\tfrac{1}{2}y^2 - \tfrac{1}{12}y^3]\Big|_0^4 = \pi[(8 - \tfrac{64}{12}) - 0] = \tfrac{8}{3}\pi.$$

c) WeBWorK

4. WeBWorK problem.

5. a) Must use disks because you can't integrate $x \ln x$. $y = \ln x \Rightarrow x = e^y$.



$$\begin{aligned} V &= \pi \int_0^1 (e)^2 - (e^y)^2\,dy = \pi \int_0^1 e^2 - e^{2y}\,dy \\ &= \pi[e^2 y - \tfrac{1}{2}e^{2y}]\Big|_0^1 = \pi[(e^2 - \tfrac{1}{2}e^2) - (-\tfrac{1}{2})] = \pi(e^2+1)/2 \end{aligned}$$

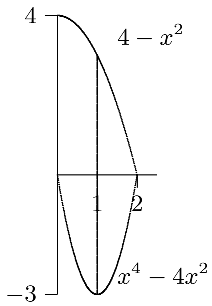
b) No help.

6. The height of each cross-sectional square is x^2. So cross-sectional area is: $A(x) = (x^2)^2 = x^4$. So

$$V = \int_0^2 A(x)\,dx = \int_0^2 x^4\,dx = \frac{1}{5}x^5\Big|_0^2 = \frac{32}{5}\ \text{cm}^3$$

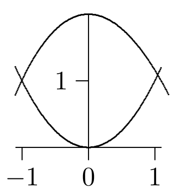
7. Use the half-angle forumlas. $V = \int_0^{\pi/4} \pi\cos^2 x - \pi \sin^2 x\,dx = \pi \int_0^{\pi/4} \frac{1}{2} + \frac{1}{2}\cos 2x - (\frac{1}{2} - \frac{1}{2}\cos 2x)\,dx = \pi \int_0^{\pi/4} \cos 2x\,dx = \frac{\pi}{2}\sin(2x)\Big|_0^{\pi/4} = \frac{\pi}{2}[1-0] = \frac{\pi}{2}.$

8. Must use shells since we can't solve for x in terms of y when $y = x^4 - 4x^2$. Intersect: $4 - x^2 = x^4 - 4x^2 \Rightarrow x^4 - 3x^2 - 4 = (x^2+1)(x^2-4) = (x^2+1)(x-2)(x+2) = 0 \Rightarrow x = 2$, not $x = -2$. Here's the figure:



$$\begin{aligned} V &= 2\pi \int_0^2 x(4-x^2) - x(x^4-4x^2)\,dx \\ &= 2\pi \int_0^2 4x + 3x^3 - x^5\,dx \\ &= 2\pi[2x^2 + \tfrac{3}{4}x^4 - \tfrac{1}{6}x^6]\Big|_0^2 \\ &= 2\pi[(8+12-\tfrac{32}{3}) - 0] = \tfrac{56}{3}\pi. \end{aligned}$$

9. Intersect: $x^2 = 2 - x^2 \Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1$. Upper minus lower.



$$\begin{aligned} V &= \pi \int_{-1}^1 (2-x^2)^2 - (x^2)^2\,dx \\ &= \pi \int_{-1}^1 4 - 4x^2\,dx = \pi[4x - \tfrac{4}{3}x^3]\Big|_{-1}^1 \\ &= \pi[(4-\tfrac{4}{3}) - (-4+\tfrac{4}{3})] = \tfrac{16}{3}\pi. \end{aligned}$$

10. The Riemann sum for volume using slices or disks is $\sum_{k=1}^n A(x_k)\Delta x$. Here the cross-sectional area will be a circle of radius r_k and from the table $\Delta x = 40$ (change in height). So each cross-section is a disk with volume $\pi r_k^2 \Delta x$. For an upper sum pick the largest of the endpoint radii values. $\pi(30)^2(40)+\pi(30)^2(40)+\pi(26)^2(40)+\pi(24)^2(40)+\pi(20)^2(40) = 138080\pi$ cu. cm. For a lower sum pick the smallest of the endpoint values. $\pi(28)^2(40)+\pi(26)^2(40)+\pi(24)^2(40)+\pi(20)^2(40)+\pi(18)^2(40) = 110400\pi$ cu. cm.

11. Use shells.

$$V = \int_r^R 2\pi x\left[\sqrt{R^2-x^2} - (-\sqrt{R^2-x^2})\right]\,dx = 2\pi \int_r^R 2x\left[\sqrt{R^2-x^2}\right]\,dx.$$

Use u-sub: $u = R^2 - x^2$, $du = -2x\,dx$, $-du = 2x\,dx$. Limits: $x = r \Rightarrow u = R^2 - r^2$, $x = R \Rightarrow u = 0$.

$$V = -2\pi \int_{R^2-r^2}^0 \sqrt{u}\,du = -\frac{4\pi}{3}u^{3/2}\Big|_{R^2-r^2}^0 = \frac{4\pi}{3}(R^2-r^2)^{3/2}.$$

12. a) Must integrate along the y-axis. $x = \tan y$ so $A = \int_0^{\pi/4} \tan y\,dy = \ln|\sec y|\Big|_0^{\pi/4} = (\ln\sqrt{2}) - 0 = \frac{1}{2}\ln 2$.

b) $V = \int_0^{\pi/4} \pi(\tan y)^2\,dy = \pi\int_0^{\pi/4} \sec^2 y - 1\,dy = \tan y - y\Big|_0^{\pi/4} = \pi(1 - \frac{1}{4}\pi)$.

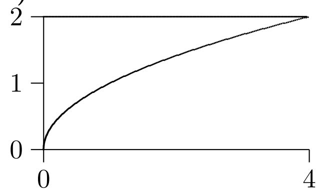
13. The cross-sectional slices perpendicular to the x-axis are semi-circles each of whose diameters the distance between $f(x)$ and $g(x)$. So the diameter is $x - (x^2-2)$ or $-x^2+x+2$. So the radius is $r = \frac{-x^2+x+2}{2}$. So the cross-sectional area is $A(x) = \text{semi-circle} = \frac{\pi}{2}\cdot\left(\frac{-x^2+x+2}{2}\right)^2 = \frac{\pi(-x^2+x+2)^2}{8}$. So the volume is

$$V = \int_{-1}^2 \frac{\pi(-x^2+x+2)^2}{8}\,dx.$$

14. a) Extra Credit

15. $V = \int_0^{150} 4y^2\,dy = \frac{4}{3}y^3\Big|_0^{150} = 4,500,000$ cu. m.

16. **a)** Outside minus inside.



$$V = \pi\int_0^4 \pi(2)^2\,dx - \int_0^4 \pi(\sqrt{x})\,dx$$
$$= \pi\int_0^4 4 - x\,dx$$
$$= \pi[4x - \tfrac{1}{2}x^2]\Big|_0^4 = \pi[(16-8)-0] = 8\pi$$

b) $V = \int_0^4 \pi(2-\sqrt{x})^2\,dx - \pi\int_0^4 \pi(2-2)^2\,dx = \pi\int_0^4 4 - 4x^{1/2} + x\,dx = \pi[4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2]\Big|_0^4$
$= \pi[(16 - \frac{64}{3} + 8) - 0] = \frac{8}{3}\pi.$

c) $V = \int_0^4 \pi(4-\sqrt{x})^2\,dx - \pi\int_0^4 \pi(4-2)^2\,dx = \pi\int_0^4 12 - 8x^{1/2} + x\,dx = \pi[12x - \frac{16}{3}x^{3/2} + \frac{1}{2}x^2]\Big|_0^4$
$= \pi[(48 - \frac{128}{3} + 8) - 0] = \frac{40}{3}\pi.$