Application: Area Between Two Curves

In this chapter we extend the notion of the area under a curve and consider the area of the region *between* two curves. To solve this problem requires only a minor modification of our point of view. We'll not need to develop any additional techniques of integration for the moment. However, we will also see that that we can think of the process used to find the area between two curves as an *accumulation process*, as we discussed earlier when we found the net distance traveled by integrating a velocity function. This theme of accumulation will be critical in the subsequent applications we carry out. Make sure you spend some time understanding this idea. Our objectives for this chapter are to

- Determine the area between two continuous curves using integration.
- Similarly, determine the area between two intersecting curves.
- Understand integration as an accumulation process.

6.1 Area of a Region Between Two Curves

With just a few modifications, we extend the application of definite integrals from finding the area of a region under a curve to finding the area of a region between two curves.

Consider two functions f and g that are continuous on the interval $[a, b]$.

In Figure 6.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f. There we can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 6.2

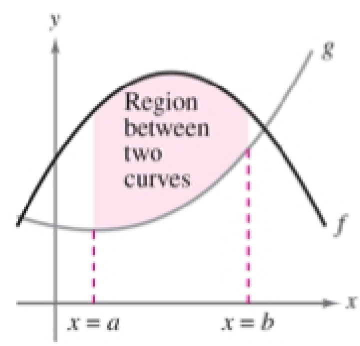


Figure 6.1: Find the area of the region between the curves f and g. (Diagram from Larson & Edwards)

The Riemann Sum Approach

Now let's step back and take a slightly different point of view on this. Remember that definite integrals are really limits of Riemann sums. So suppose we use a regular partition of $[a, b]$ into n equal subintervals of width Δx. We use the partition to subdivide the region between the two curves into n rectangles. We won't draw all of them, but rather we will draw a single *representative rectangle* (see Figure 6.3). The width of the rectangle is Δx and the height is $f(x_i) - g(x_i)$ where x_i is the right-hand endpoint of the ith subinterval.

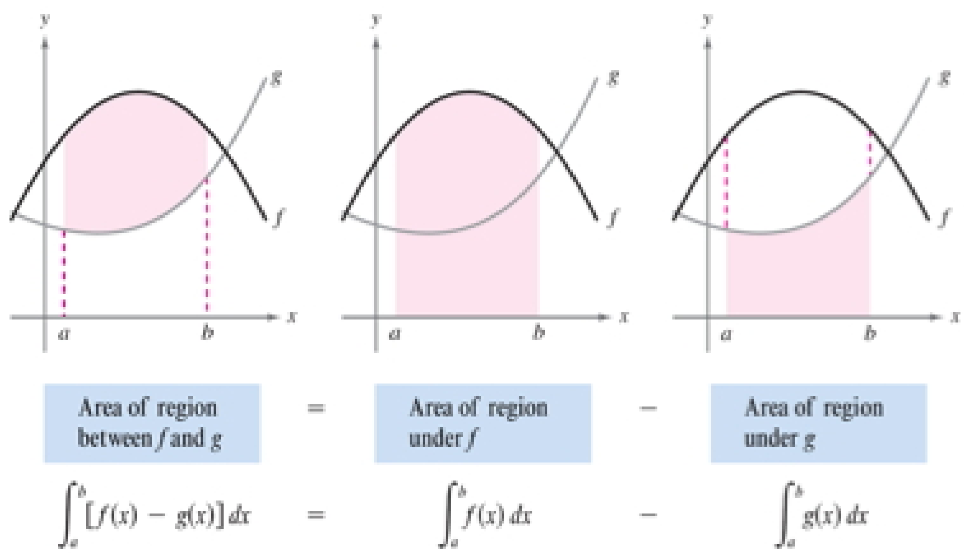


Figure 6.2: Find the area of the region between the curves f and g when both f and g lie above the x-axis and g lies below f. (Diagram from Larson & Edwards)

The area of the representative rectangle is

$$\text{height} \times \text{width} = [f(x_i) - g(x_i)]\Delta x.$$

We add up all the n rectangles to get an approximation to the total area between the curves:

$$\text{Approximate Area beween } f \text{ and } g = \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

To improve the approximation we take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

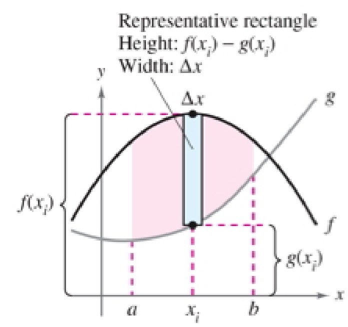


Figure 6.3: The area of the ith rectangle is $[f(x_i) - g(x_i)]\Delta x$. (Diagram from Larson & Edwards).

Now because both f and g are *continuous* we know that this limit exists and, in fact, equals a definite integral. Thus, the area of the given region is

$$\text{Area beween } f \text{ and } g = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)]\,dx.$$

Let's summarize what we have found in a theorem.

THEOREM 6.1. If f and g are continuous on $[a,b]$ and $g(x) \le f(x)$ for all x in $[a,b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$\boxed{\text{Area beween } f \text{ and } g = \int_a^b [f(x) - g(x)]\,dx.}$$

Note: This area will always be non-negative.

Notice that the theorem gives the same answer as our earlier geometric argument in Figure 6.2 However, unlike in Figure 6.2, notice that the theorem does not say that both curves have to lie *above* the x-axis. The same integral

$$\int_a^b [f(x) - g(x)]\,dx$$

works as long as f and g are continuous on $[a,b]$ and $g(x) \leq f(x)$ for all x in the interval $[a,b]$. The reason this same integral remains valid when one or both curves dip below the x-axis is illustrated in Figure 6.4. The height of a representative rectangle is always $f(x) - g(x)$. This is the advantage of using Riemann sums and representative rectangles. It gives us a more general argument than a simple geometric one in this case.

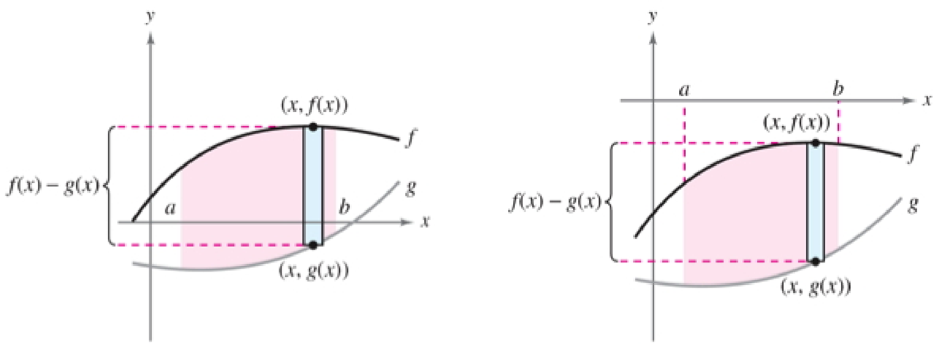


Figure 6.4: The height of a representative rectangle is $f(x) - g(x)$ whether or not one or both curves lie above or below the x-axis. (Diagram from Larson & Edwards)

Tip for Success

We will continue to use representative rectangles as we develop further applications. Drawing a figure with such representative rectangles will help you to write out the correct integral in these applications.

6.2 *Examples*

We now take a look at several examples.

EXAMPLE 6.1. Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x^3$ and the vertical lines $x = -1$ and $x = 1$.

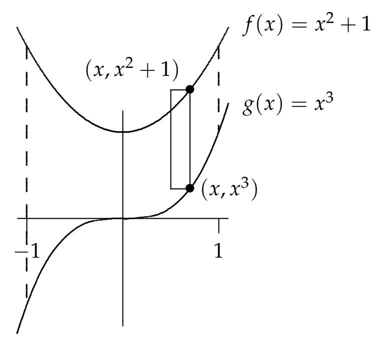


Figure 6.5: The area between f and g with a representative rectangle.

SOLUTION. After quickly plotting the graphs we see that $x^2 + 1$ lies above x^3 on the interval. So let $f(x) = x^2 + 1$ and $g(x) = x^3$. Since both are continuous (polynomials) Theorem 6.1 applies and we have

$$\begin{aligned}\text{Area beween } f \text{ and } g = \int_a^b [f(x) - g(x)]\,dx &= \int_{-1}^1 [(x^2+1) - x^3]\,dx \\ &= \frac{x^3}{3} + x - \frac{x^4}{4}\bigg|_{-1}^{1} \\ &= \left(\frac{1}{3} + 1 - \frac{1}{4}\right) - \left(-\frac{1}{3} - 1 - \frac{1}{4}\right) = \frac{8}{3}.\end{aligned}$$

Area Enclosed by Two Intersecting Curves

In Example 6.1 we found the area below one curve but above another curve on a given interval. A more common problem is a slight variation on this. Find the region enclosed by two intersecting curves. Usually the points of intersection are not provided and that becomes the first step in solving such a problem.

EXAMPLE 6.2 (Two Intersecting Curves). Find the area of the region enclosed by the graphs of $y = x^2 - 2$ and $y = x$. (In a typical problem, not even the graph is given.)

SOLUTION. Let $f(x) = x^2 - 2$ and $g(x) = x$. First we find the intersections of the two graphs:

$$x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2.$$

Which curve lies above the other on the interval $[-1, 2]$? We can test an intermediate point. The point $x = 0$ is convenient: Notice $f(0) = -2$ and $g(0) = 0$. Or we can can quickly plot the graphs (see Figure 6.6) and see that x lies above $x^2 - 2$ on the interval $[-1, 2]$. Since both are continuous (polynomials) Theorem 6.1 applies and we have (notice that g is 'on top').

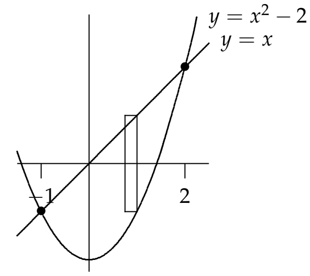


Figure 6.6: The area enclosed by $y = x^2 - 2$ and $y = x$ with a representative rectangle.

$$\begin{aligned} \text{Area enclosed by } g \text{ and } f = \int_a^b [g(x) - f(x)]\,dx &= \int_{-1}^{2} [x - (x^2 - 2)]\,dx \\ &= \frac{x^2}{2} - \frac{x^3}{3} + 2x \Big|_{-1}^{2} \\ &= \left(2 - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) = \frac{9}{2}. \end{aligned}$$

EXAMPLE 6.3 (Division into Two Regions). Find the area of the region enclosed by the graphs of $y = x^3$ and $y = x$.

SOLUTION. Let $f(x) = x^3$ and $g(x) = x$. First we find the intersections of the two graphs:

$$\begin{aligned} x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 &\Rightarrow x(x+1)(x-1) = 0 \\ &\Rightarrow x = -1, 0, 1. \end{aligned}$$

Since there are three points of intersection, we need to determine which curve lies above the other on each subinterval. On $[-1, 0]$, we can test an intermediate point $x = -\frac{1}{2}$: $f(-\frac{1}{2}) = -\frac{1}{8}$ and $g(-\frac{1}{2}) = -\frac{1}{2}$. So f lies above g. On $[0, 1]$, we test at the intermediate point $x = \frac{1}{2}$: $f(\frac{1}{2}) = \frac{1}{8}$ and $g(\frac{1}{2}) = \frac{1}{2}$. So g lies above f. Also we can quickly plot the graphs (see Figure 6.7) and the same behavior. Since both are continuous (polynomials) Theorem 6.1 applies. However, we will have to split the integration into two pieces since the top and bottom curves change at the point $x = 0$ in the interval $[-1, 1]$.

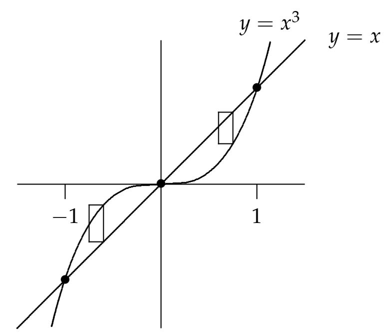


Figure 6.7: The area enclosed by $y = x^3$ and $y = x$. The top and bottom curve switch at $x = 0$. There are two different representative rectangles.

$$\begin{aligned} \text{Area enclosed by } g \text{ and } f &= \int_{-1}^{0} [x^3 - x]\,dx + \int_0^1 [x - x^3]\,dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 \\ &= \left([0] - \left[\frac{1}{4} - \frac{1}{2}\right]\right) - \left(\left[\frac{1}{2} - \frac{1}{4}\right] - [0]\right) = \frac{1}{2}. \end{aligned}$$

EXAMPLE 6.4. Find the area of the region enclosed by the graphs of $y = x\sqrt{x+1}$ and $y = 2x$.

SOLUTION. Let $f(x) = x\sqrt{x+1}$ and $g(x) = 2x$. First we find the intersections of the two graphs:

$$\begin{aligned} x\sqrt{x+1} = 2x \Rightarrow x^2(x+1) = 4x^2 \Rightarrow x^3 - 3x^2 = 0 &\Rightarrow x^2(x-3) = 0 \\ &\Rightarrow x = 0, 3. \end{aligned}$$

To determine which curve lies above the other on On $[0, 3]$, we can test an intermediate point, say $x = 1$: $f(1) = \sqrt{2}$ and $g(1) = 2$. So g lies above f. We can quickly plot the graphs (see Figure 6.8). Since both are continuous Theorem 6.1 applies.

$$\text{Area enclosed by } g \text{ and } f = \int_0^3 [2x - x\sqrt{x+1}]\,dx = \int_0^3 2x\,dx - \int_0^3 x\sqrt{x+1}\,dx.$$

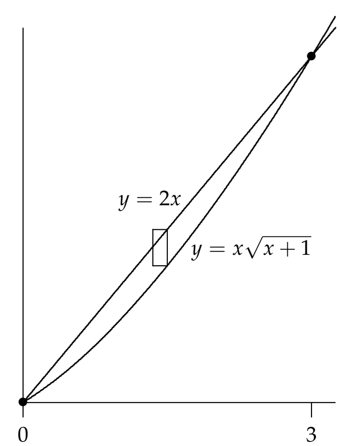


Figure 6.8: The area enclosed by $y = x\sqrt{x+1}$ and $y = 2x$ and a representative rectangle.

For the second integral we use the substitution

$$u = \sqrt{x+1} \Rightarrow u^2 = x+1 \Rightarrow u^2 - 1 = x \Rightarrow 2u\,du = dx$$

and change the limits:

$$\text{when } x = 0,\ u = \sqrt{0+1} = 1;\ \text{when } x = 3,\ u = \sqrt{3+1} = 2.$$

So

$$\begin{aligned}
\int_0^3 2x\,dx - \int_0^3 x\sqrt{x+1}\,dx &= \int_0^3 2x\,dx - \int_1^2 (u^2-1)\cdot u \cdot 2u\,du \\
&= x^2\Big|_0^3 - \int_1^2 2u^4 - u^2\,du \\
&= (9-0) - \left(\frac{2u^5}{5} - \frac{u^3}{3}\right)\Big|_1^2 \\
&= 9 - \left(\left[\frac{64}{5} - \frac{8}{3}\right] - \left[\frac{1}{5} - \frac{1}{3}\right]\right) = \frac{19}{15}.
\end{aligned}$$

Variations

Here are some additional 'variations on the theme' of Theorem 6.1.

EXAMPLE 6.5 (Multiple Curves, Multiple Regions). Find the area of the region enclosed by the graphs of $y = 8 - x^2$, $y = 7x$, and $y = 2x$ in the first quadrant.

SOLUTION. This time there are three curves to contend with. Since the curves are relatively simple (an upside-down parabola and two lines through the origin, it is relatively easy to make a sketch of the region. See Figure 6.9. Let $f(x) = 8 - x^2$, $g(x) = 7x$, and $h(x) = 2x$. A wedge-shaped region is determined by all three curves. Notice that the 'top' curve of the region switches from $g(x)$ to $f(x)$. We find the intersections of the pairs of graphs:

$f(x) = g(x) \Rightarrow 8 - x^2 = 7x \Rightarrow x^2 + 7x - 8 = 0 \Rightarrow (x-1)(x+8) = 0 \Rightarrow x = 1$ (not -8).

$f(x) = h(x) \Rightarrow 8 - x^2 = 2x \Rightarrow x^2 + 2x - 8 = 0 \Rightarrow (x-2)(x+4) = 0 \Rightarrow x = 2$ (not -4).

$g(x) = h(x) \Rightarrow 7x = 2x \Rightarrow 5x = 0 \Rightarrow x = 0.$

The region is thus divided into two subregions and the graph gives the relative positions of the curves. Since all the functions are continuous Theorem 6.1 applies.

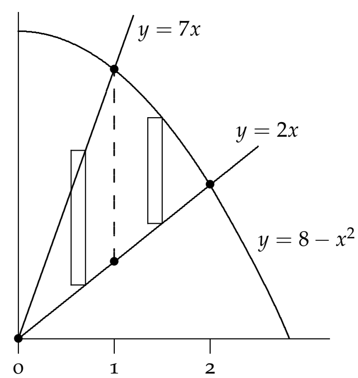
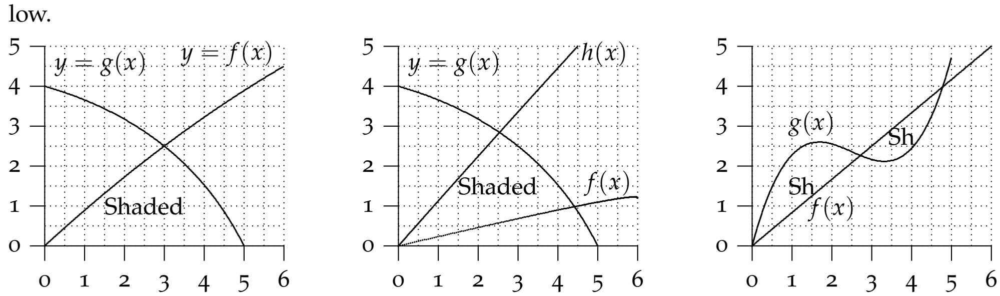


Figure 6.9: The area enclosed by $y = 8 - x^2$, $y = 7x$, and $y = 2x$ in the first quadrant. There are two representative rectangles because the top curve changes.

$$\begin{aligned}
\text{Area enclosed by } f, g, \text{ and } h &= \int_0^1 [7x - 2x]\,dx + \int_1^2 [(8-x^2) - 2x\,dx \\
&= \int_0^1 [5x]\,dx + \left(8x - \frac{x^3}{3} - x^2\right)\Big|_1^2 \\
&= \left(\frac{5x^2}{2}\right)\Big|_0^1 + \left(\left[16 - \frac{8}{3} - 4\right] - \left[8 - \frac{1}{3} - 1\right]\right) \\
&= \left(\frac{5}{2} - 0\right) + \left(\frac{8}{3}\right) = \frac{31}{6}.
\end{aligned}$$

YOU TRY IT 6.1. Set up the integrals using the functions $f(x)$, $g(x)$, and $h(x)$ and their points of intersection that would be used to find the shaded areas in the three regions below.



YOU TRY IT 6.2. Sketch the regions for each of the following problems before finding the areas.

(*a*) Find the area enclosed by the curves $y = x^3$ and $y = x^2$. (Answer: 1/12)

(*b*) Find the area enclosed by the curves $y = x^3 + x$ and $y = 3x^2 - x$. (Answer: 1/2)

(*c*) Find the area between the curves $f(x) = \cos x + \sin x$ and $g(x) = \cos x - \sin x$ over $[0, 2\pi]$. (Answer: 8)

YOU TRY IT 6.3. Sketch each region before finding its area:

(*a*) The area in the first quadrant enclosed by $y = \cos x$, $y = \sin x$, and the y axis. (Answer: $\sqrt{2} - 1$)

(*b*) The area enclosed by $y = x^3$ and $y = \sqrt[3]{x}$. (Answer: 1)

(*c*) The area enclosed by $y = x^3 + 1$ and $y = (x+1)^2$. (Answer: 37/12)

(*d*) Harder integration: The area enclosed by $y = x\sqrt{2x+3}$ and $y = x^2$. (Answer: $\frac{6}{5}\sqrt{3} + \frac{26}{15}$.)

EXAMPLE 6.6. Find the area of the region in the first quadrant enclosed by the graphs of $y = 1$, $y = \ln x$, and the x- and y-axes.

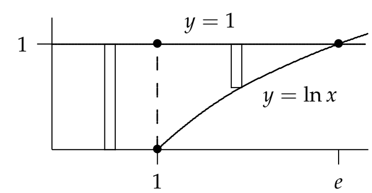


Figure 6.10: in the first quadrant enclosed by the graphs of $y = 1$, $y = \ln x$, and the x- and y-axes. There are two representative rectangles because the bottom curve changes.

SOLUTION. It is easy to sketch the region. See Figure 6.10. The curve $y = \ln x$ intersects the x-axis at $x = 1$ and the line $y = 1$ at $x = e$. Notice that the 'bottom' curve of the region switches from x-axis to $y = \ln x$ at $x = 1$. The region is divided into two subregions (one is a square!) and the graph gives the relative positions of the curves. Since both the functions are continuous Theorem 6.1 applies.

$$\text{Area} = \int_0^1 1\,dx + \int_1^e 1 - \ln x\,dx$$

We can rewrite the integral in a more convenient way. Notice that the area the we are trying to find is really just the rectangle of height 1 minus the area under $y = \ln x$ on the interval $[1, e]$. (Yet another way of saying this is that we are splitting $\int_1^e 1 - \ln x\,dx$ into two integrals $\int_1^e 1\,dx$ and $\int_1^e -\ln x\,dx$ and then combining the two integrals $\int_0^1 1\,dx + \int_1^e 1\,dx$ into one leaving $-\int_1^e \ln x\,dx$.) We get

$$\text{Area} = \int_0^e 1\,dx - \int_1^e \ln x\,dx = e + ???$$

The problem is that we do not know an antiderivative for $\ln x$. So we need another way to attack the problem. We describe this below.

6.3 *Point of View: Integrating along the y-axis*

Reconsider Example 6.6 and change our point of view. Suppose that we drew our representative rectangles horizontally instead of vertically as in Figure 6.11. The integration now takes place along the y-axis on the interval $[0, 1]$. Using inverse functions, the function $y = \ln x$ is viewed as $x = g(y) = e^y$. Now the 'width' of a representative rectangle is Δy and the (horizontal) 'height' of the ith such rectangle is given by $g(y_i)$.

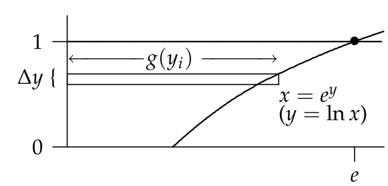


Figure 6.11: The region in the first quadrant enclosed by the graphs of $y = 1$, $y = \ln x$, and the x- and y-axes. There are two representative rectangles because the bottom curve changes.

As we saw earlier in the term with integration along the x-axis, since g is continuous, the exact area of the region is given by

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} g(y_i)\Delta y = \int_c^d g(y)\,dy.$$

In our particular case, the interval $[c, d] = [0, 1]$ along the y-axis. The function $g(y) = e^y$. So the area of the region is in Figure 6.11 (or equivalently 6.10) is

$$\text{Area} = \int_c^d g(y)\,dy = \int_0^1 e^y\,dy = e^y\Big|_0^1 = e - 1.$$