The Fundamental Theorems of Calculus

The Fundamental Theorem of Calculus, Part II

Recall the *Take-home Message* we mentioned earlier. Example 1.0.10 points out that even though the definite integral 'solves' the area problem, we must still be able to evaluate the Riemann sums involved. If the region is not a familiar one and we can't determine

$$\lim_{\text{all } \Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

then we are stuck in trying to evaluate $\int_a^b f(x)\,dx$. In other words, *we must find another method to evaluate definite integrals.* We now make the connection between antiderivatives and definite integrals. To do this, we will need to use the Mean Value Theorem in the following form:

THEOREM 1.7 (MVT: The Mean Value Theorem)**.** Assume that

1. F is continuous on the closed interval $[x_{k-1}, x_k]$;
2. F is differentiable on the open interval (x_{k-1}, x_k);

Then there is some point c_k between x_{k-1} and x_k so that

$$F'(c_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}.$$

This is equivalent to saying $F(x_k) - F(x_{k-1}) = F'(c_k) \cdot (x_k - x_{k-1})$. Or using the notation of Riemann sums,

$$F(x_k) - F(x_{k-1}) = F'(c_k)\Delta x_k.$$

THEOREM 1.8 (FTC Part II). Assume that f is continuous on $[a, b]$ and that F is an antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Note: We will cover what your text calls Part I of the FTC shortly.
Recall: F is an antiderivative of f means that $F' = f$ on $[a, b]$.

Before we do the proof, let's look at an example so you can appreciate what this theorem says.

EXAMPLE 1.0.16. Let $f(x) = x^2$ on $[0, 2]$. An antiderivative is $F(x) = \frac{1}{3}x^3$. So Theorem 1.8 says

$$\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}.$$

Wow! That's a heck of a lot simpler than doing a limit of Riemann sums. Now to 'pay' for this convenience, we need to spend a few minutes working through the proof of the theorem. But it will pay big dividends.

Proof. Use a regular partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ into n equal-width subintervals. So $x_k - x_{k-1} = \Delta x$. Now F is an antiderivative of f means that $F' = f$. Therefore F is differentiable (and hence continuous) on $[a, b]$ and each of its subintervals

$[x_{k-1}, x_k]$. So the MVT (Theorem 1.7) applies to each subinterval, as indicated below. Then add the results:

$$\begin{array}{rlllll}
\text{On } [x_0, x_1]: & F(x_1) - F(x_0) & = & F'(c_1)\Delta x & = & f(c_1)\Delta x \\
\text{On } [x_1, x_2]: & F(x_2) - F(x_1) & = & F'(c_2)\Delta x & = & f(c_2)\Delta x \\
\text{On } [x_2, x_3]: & F(x_3) - F(x_2) & = & F'(c_3)\Delta x & = & f(c_3)\Delta x \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{On } [x_{n-2}, x_{n-1}]: & F(x_{n-1}) - F(x_{n-2}) & = & F'(c_{n-1})\Delta x & = & f(c_{n-1})\Delta x \\
\text{On } [x_{n-1}, x_n]: & F(x_n) - F(x_{n-1}) & = & F'(c_n)\Delta x & = & f(c_n)\Delta x \\
\hline
& F(x_n) - F(x_0) & = & & & \sum_{k=1}^{n} f(c_k)\Delta x
\end{array}$$

We see that the sum in the first column 'telescopes' because all of the terms cancel except the last and first. Since $x_0 = a$ and $x_n = b$, we can rewrite equation () as

$$F(b) - F(a) = \sum_{k=1}^{n} f(c_k)\Delta x.$$

Taking the limit of both sides

$$\lim_{n\to\infty} (F(b) - F(a)) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(c_k)\Delta x \right)$$

and using the fact that f is continuous so it is integrable (Theorem 1.3), we get

$$F(b) - F(a) = \int_a^b f(x)\, dx.$$

Amazing! □

EXAMPLE 1.0.17. Find the area under $f(x) = -x^2 + 4x - 3$ on $[1, 3]$.

SOLUTION. We did this with Riemann sums in Example 1.0.4 (see Figure 1.18). But now it is easy using FTC II and antiderivatives

$$\text{Area} = \int_1^3 -x^2 + 4x - 3\, dx = -\frac{x^3}{3} + 2x^2 - 3x \Big|_1^3 = (-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3 \right) = \frac{4}{3},$$

which is the answer we got earlier after an entire page of calculations!

EXAMPLE 1.0.18. Recall that in Example 1.0.10 we were unable evaluate $\int_0^\pi \sin x\, dx$ because we could not simplify the corresponding Riemann sum. Now, however, using FTC II

$$\int_0^\pi \sin x\, dx = -\cos x \Big|_0^\pi = -\cos \pi - (-\cos 0) = 1 + 1 = 2.$$

Further,

$$\int_0^{2\pi} \sin x\, dx = -\cos x \Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = -1 + 1 = 0$$

just as we saw in Example 1.0.9 and Figure 1.26.

EXAMPLE 1.0.19. Evaluate $\int_0^2 e^{4x}\, dx$. Using FTC II

$$\int_0^2 e^{4x}\, dx = \frac{1}{4} e^{4x} \Big|_0^2 = \frac{1}{4} e^8 - \frac{1}{4} e^0 = \frac{1}{4} \left(e^8 - 1 \right).$$

EXAMPLE 1.0.20. Evaluate $\int_1^9 \sqrt{3x}\, dx$. Using FTC II

$$\int_3^{12} \sqrt{3x}\, dx = \frac{1}{3} \cdot \frac{2}{3} \cdot (3x)^{3/2} \Big|_1^{12} = \frac{2}{9} (6 - 3) = \frac{2}{3}.$$

EXAMPLE 1.0.21. Evaluate $\int_1^2 \frac{4x^2+1}{x}\,dx$. Using FTC II

$$\int_1^2 \frac{4x^2+1}{x}\,dx = \int_1^2 4x + \frac{1}{x}\,dx = 2x^2 + \ln|x|\Big|_1^2 = (8+\ln 2) - (2+0) = 6 + \ln 2.$$

EXAMPLE 1.0.22. Return to the unit semi-circle problem: Evaluate $\int_{-1}^1 \sqrt{1-x^2}\,dx$. Since the region is a unit semi-circle, we know

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$

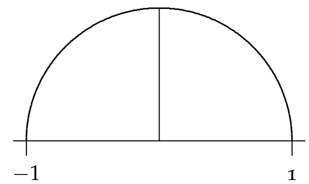


Figure 1.34: The area between $f(x) = \sqrt{1-x^2}$ and the x-axis is a semi-circle above the axis.

However, we still do not know an antiderivative of $\sqrt{1-x^2}$. This highlights the hypothesis in FTC II. You must know an anti-derivative F of f to be able to use the theorem. Later in the term we will spend a fair amount of time on different techniques of antidifferentiation. In this way, FTC II becomes truly useful. In the process we will find an antiderivative of $f(x) = \sqrt{1-x^2}$.

EXAMPLE 1.0.23 (Net Area vs. Total Area). Let $f(x) = x^3 - 3x^2$ on $[0,4]$.

(1) Find the *net* area between f and the x-axis.

(2) Find the *total* area enclosed by f and the x-axis (this means all regions count as positive area).

SOLUTION. (1) To determine the net area we just evaluate $\int_0^4 x^3 - 3x^2\,dx$. Applying FTC II, we find

$$\int_0^4 x^3 - 3x^2\,dx = \frac{x^4}{4} - x^3\Big|_0^4 = (64-64) - (0-0) = 0.$$

In other words, the net area is 0, so the areas above and below the x-axis must exactly cancel each other out.

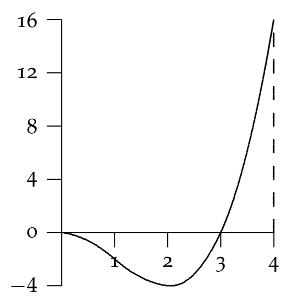
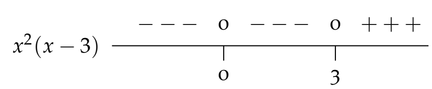


Figure 1.35: The areas between $f(x) = x^3 - 3x^2$ the x-axis on $[0,4]$ above and below the axis cancel each other.

(2) To determine the total area enclosed, we need to know where $f(x)$ is positive and where it is negative.

$$x^3 - 3x^2 = x^2(x-3) = 0 \Rightarrow x = 0, 3.$$



The number line to the right shows that $x^3 - 3x^2 \le 0$ on $[0,3]$ and $x^3 - 3x^2 \ge 0$ on $[3,4]$. We can now find the total area in a couple of different ways. The simplest conceptually is to split the interval into two pieces, changing the sign of the first piece, since the net area is negative there.

$$\begin{aligned}\text{Total Area} &= -\int_0^3 x^3 - 3x^2\,dx + \int_3^4 x^3 - 3x^2\,dx \\ &= -\left[\frac{x^4}{4} - x^3\Big|_0^3\right] + \left[\frac{x^4}{4} - x^3\Big|_3^4\right] \\ &= \left[\left(\frac{81}{4} - 27\right) - (0-0)\right] + \left[(64-64) - \left(\frac{81}{4} - 27\right)\right] = \frac{27}{2}.\end{aligned}$$

A second way of conceptualizing the problem is to change the function to $|f(x) = |x^3 - 3x^2|$ which is always non-negative. Our earlier work determining where f was positive and negative shows that

$$|x^3 - 3x^2| = \begin{cases} x^3 - 3x^2, & \text{if } x \ge 3\,, \\ -x^3 + 3x^2 & x \le 3. \end{cases}$$

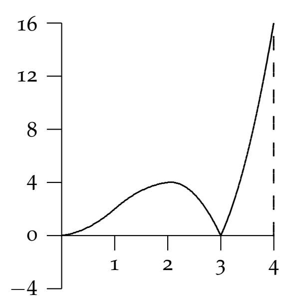


Figure 1.36: The graph of $|f(x)| = |x^3 - 3x^2|$. Compare to Figure 1.35.

So

$$\text{Total Area} = -\int_0^4 |x^3 - 3x^2|\,dx = \int_0^3 -x^3 + 3x^2\,dx + \int_3^4 x^3 - 3x^2\,dx$$

which is equivalent to the first method. The choice is yours. In either case, to find the total area, you must first determine where the integrand $f(x)$ is positive and where it is negative.

EXAMPLE 1.0.24. Evaluate $\int_0^2 x^3 - 2x\,dx$. Applying FTC II, we find

$$\int_0^2 x^3 - 2x\,dx = \frac{x^4}{4} - x^2\Big|_0^2 = (4-4) - (0-0) = 0.$$

In other words, the net area is 0, so the areas above and below the x-axis must cancel each other out.

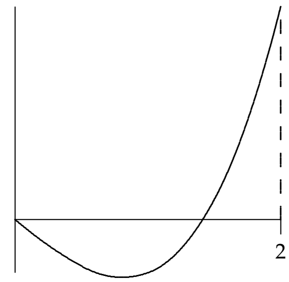


Figure 1.37: The areas enclosed by $f(x) = x^3 - 2x$ above and below the x-axis on $[0,2]$ cancel each other.

EXAMPLE 1.0.25. Evaluate $\int_{-3}^{3} |2x+2|\,dx$.

SOLUTION. Take a look at the graph of the function. We don't have an antiderivative of $f(x) = |2x+2|$ on $[-3,3]$. However,

$$|2x+2| = \begin{cases} 2x+2 & \text{if } x \geq -1, \\ -2x-2 & \text{if } x < -1. \end{cases}$$

Now we can use the additivity of the definite integral (Theorem 1.4) and split the integral into two pieces and apply FTC II.

$$\begin{aligned} \int_{-3}^{3} |2x+2|\,dx &= \int_{-3}^{-1} -2x-2\,dx + \int_{-1}^{3} 2x+2\,dx \\ &= -x^2 - 2x\Big|_{-3}^{-1} + x^2 + 2x\Big|_{-1}^{3} \\ &= [(-1+2) - (-9+6)] + [(9+6) - (1-2)] \\ &= 20. \end{aligned}$$

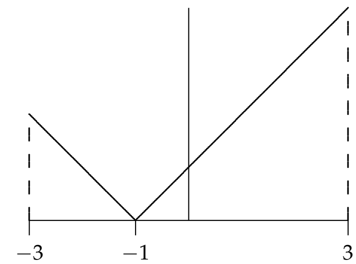


Figure 1.38: The area enclosed by $f(x) = |2x+2|$ on $[-3,3]$ is split into two pieces.

1.1 The FTC and Riemann Sums. An Application of Definite Integrals: Net Distance Travelled

In the next few sections (and the next few chapters) we will see several important applications of definite integrals. When first taking calculus it is easy to confuse the integration (with its Riemann sums) process with simple 'antidifferentiation.' While the First Fundamental Theorem connects these two, they are not the same thing. Most important, though, the determination of many quantities can be approximated (interpreted) as Riemann sums and hence evaluated as definite integrals *even though* it is not obvious at the outset that antidifferention should be involved. The Riemann sum part turns out to be critical. Here's an example of what I mean.

Suppose we know that the velocity of an object traveling along a line (think car on a straight highway) is given by a continuous function $v(t)$, where t represents time on the interval $[a,b]$. How might we determine the net distance the object has travelled? Well, we know that if the velocity were *constant*, then

$$\text{distance} = \text{rate} \times \text{time}.$$

Observe: Distance has been expressed as product, much the way we assumed earlier that the area of a rectangle could be expressed as a product:

$$\text{area of a rectangle} = \text{height} \times \text{base}.$$

We can extend this analogy to Riemann sums and area under curves. While the velocity is *not constant on long intervals* since the velocity is continuous it is *nearly constant on short time intervals*. So divide the time interval using a regular partition $\{t_o, t_1, t_2, \ldots, t_n\}$ of n subintervals of length Δt. Next, pick any point in the kth subinterval (we might as well choose the right-hand endpoint t_k for convenience) and evaluate the velocity $v(t_k)$ there. Then the distance traveled during the kth time interval approximated as

$$\text{distance} = \text{rate} \times \text{time} \approx v(t_k) \times \Delta t.$$

Since the net distance travelled is the sum of the distances traveled on each subinterval which is approximately

$$\text{Net Distance} \approx \sum_{k=1}^{n} v(t_k) \times \Delta t.$$

The approximation is improved by letting n get large and taking a limit.

$$\boxed{\text{Net Distance} = \lim_{n\to\infty} \sum_{k=1}^{n} v(t_k) \times \Delta t = \int_a^b v(t)\,dt.} \tag{1.8}$$

Since v was assumed to be continuous, then by Theorem 1.3 we know that the limit exists and can be evaluated as a definite integral using antidifferentiation assuming we know an appropriate antiderivative. Finally, think about how we interpreted definite integrals geometrically: as (net) area under a curve. What we have just shown is that the net distance travelled over the time interval $[a, b]$ is just the *net area under the velocity curve*. That's not obvious at first. But being able to

What's your point? The key point here is that we were able to use a 'divide and conquer' process to determine the velocity. Let's list it as a series of steps.

- We subdivided the quantity into small bits,
- and we were able to approximate the each bit as a *product*.
- When we reassembled (summed) the bits, we found we had a Riemann sum.
- Once we had a Riemann sum we could take a limit as the number of bits got large.
- The limit was a definite integral
- which we could evaluate easily (if we know an antiderivative) using the First Fundamental Theorem of Calculus.

We will use this process repeatedly over the next few weeks. Look for it in other courses. What quantities do you know are 'products'? What about the amount of electricity used in your home? If you know the *flow rate* of electricity into your house (go look at your electric meter spinning around), then the *amount* of electricity consumed can be computed as an integral, just as we did with velocity (rate) and distance.

EXAMPLE 1.1.1. Ok, we better do one example. If the velocity of an object moving along a straight line is given by $v(t) = 2t + 3\sin t$ m/s on the interval $[0, \pi]$. Find the net distance travelled.

SOLUTION. Ok, we just need to use (1.8).

$$\text{Net Distance} = \int_a^b v(t)\,dt = \int_0^\pi 2t + 3\sin t\,dt$$

$$= t^2 - 3\cos t\Big|_0^\pi = (\pi^2 + 3) - (0 - 3) = \pi^2 + 6 \text{ m}.$$

YOU TRY IT 1.10. If the velocity of an object moving along a straight line is given by $v(t) = \frac{1}{t} + \sqrt{t}$ m/s on the interval $[1, 4]$. Find the net distance travelled. (Answer: $\ln 4 - \frac{14}{3}$ m.)