1.2 *Accumulation Functions: The Definite Integral as a Function*

When we compute a definite integral $\int_a^b f(x)\,dx$ we get a number which we may interpret as the net area between f and the x-axis. If we take this idea and let the upper limit vary, then we can define a so-called 'accumulation function.'

Let $f(x)$ be continuous (so it is integrable) on $[a,b]$. For any number x in $[a,b]$ define

$$A(x) = \text{'net area between } f \text{ and the } x\text{-axis from } a \text{ to } x\text{'}$$

or more precisely, define

$$A(x) = \int_a^x f(t)\,dt. \tag{1.9}$$

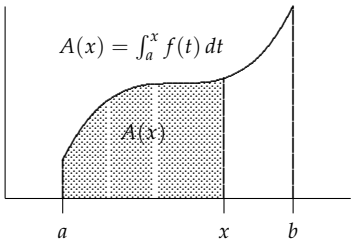

Figure 1.39: Area 'accumulates' as x moves from a to b.

Now $A(x)$ is a complicated function, but it is a function. For each x in $[a,b]$, we get an output number, namely $\int_a^x f(t)\,dt$. Notice that the variable of integration is t here. We cannot use x because x represents the right endpoint of the interval of integration.

Even though $A(x)$ may be complicated notice that if $f(t)$ actually were the *velocity* $v(t)$ of an object, then $A(x) = \int_a^x v(t)\,dt$ would represent the *displacement* of the object. More precisely, $A(x)$ would represent the net change in the position from time a to time x. In other words $A(x)$ would be the net distance that had been traveled. For the moment, let's take a look a at couple of examples.

EXAMPLE 1.2.1. Let $f(t) = 6e^{2t}$. Then f is continuous (everywhere) so we can form the accumulation function

$$A(x) = \int_0^x 6e^{2t}\,dt = 6 \cdot \frac{1}{2} \cdot e^{2t}\Big|_0^x = 3(e^{2x} - 1).$$

Notice that the answer is a *function* because the variable x is the upper endpoint of the integral. If necessary, we could evaluate this function for various values of x.

EXAMPLE 1.2.2. Now compare Example 1.2.1 with the following situation. Let $f(t) = \sin(t^2)$. Again f is continuous (everywhere) so we can form the accumulation function

$$A(x) = \int_0^x \sin(t^2)\,dt = \text{??!?}$$

Since we don't know an antiderivative for $\sin(t^2)$ we can't proceed any further. We know (theoretically) that the accumulation function exists, but we can't determine a formula for $A(x)$. We are stuck.

However—and amazingly—even when we don't have a formula for $A(x)$ as in Example 1.2.2, we can still find the derivative of $A(x)$. Here's how.

Assume that $f(t)$ is continuous on $[a,b]$ and that $A(x) = \int_a^x f(t)\,dt$. Find $A'(x)$. By definition,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}.$$

Substituting in the definition of A, we get

$$A'(x) = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}.$$

Using the additivity of the integral (Theorem 1.4) this simplifies to (see Figure 1.40).

$$A'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}.$$

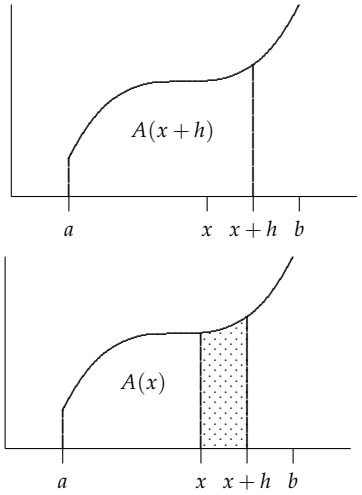

Figure 1.40: $A(x+h) - A(x)$ is the area of the shaded strip. It is nearly a rectangle in shape.

Now $\int_x^{x+h} f(t)\,dt$ is just the area of a strip between x and $x+h$. This strip is nearly a rectangle (see Figure 1.41) of height $f(x+h)$ and width h. This rectangular approximation becomes better as h gets smaller. The area of the rectangle is $f(x+h)\cdot h$. Consequently,

$$A'(x) = \lim_{h\to 0} \frac{f(x+h)\cdot h}{h} = \lim_{h\to 0} f(x+h).$$

Finally, because f is continuous, we can evaluate the limit

$$A'(x) = \lim_{h\to 0} f(x+h) = f(x).$$

Since $A(x)$ is really just $\int_a^x f(t)\,dt$, another way of writing $A'(x)$ is

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Your text calls this the first part of the Fundamental Theorem of Calculus (FTC I). We won't use it as often as FTC II, but it is useful. It is customary to state the result more carefully using $F(x)$ instead of Ax).

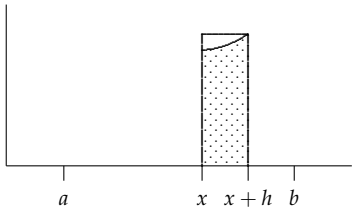

Figure 1.41: $A(x+h) - A(x)$ is approximately a rectangle (compare the shaded strip to the rectangle) with area $f(x+h)\cdot h$.

THEOREM 1.9 (Fundamental Theorem of Calculus, Part I: **FTC I**). Assume that f is continuous on an open interval I containing a. For any x in I, define

$$A(x) = \int_a^x f(t)\,dt.$$

Then

$$A'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Notice that $A'(x) = f(x)$, not $f(t)$. The variable is now x, the endpoint of the interval.

Step back for a second and think about what we have just shown. If we integrate a continuous function $f(t)$ over an interval $[a,x]$ where we think of the upper endpoint as variable, then we get a function $A(x)$ whose derivative is is $f(x)$, even if we can't figure out what $A(x)$ is! If we return to Example 1.2.2 and let $f(t) = \sin(t^2)$. Since f is continuous (everywhere) we can define the new function

$$A(x) = \int_0^x \sin(t^2)\,dt.$$

We were unable to figure out a formula for $F(x)$. However, FTC I says $A'(x) = \sin(x^2)$. Here are a couple of more examples

EXAMPLE 1.2.3. $\frac{d}{dx}\left[\int_0^x e^{t^3}\,dt\right] = e^{x^3}$ because e^{t^3} is continuous so FTC I applies. Similarly $\frac{d}{dx}\left[\int_1^x \sqrt{t^4+1}\,dt\right] = \sqrt{x^4+1}$ because $\sqrt{t^4+1}$ is continuous so FTC I applies.

EXAMPLE 1.2.4. This time suppose we have $\frac{d}{dx}\left[\int_x^0 t\sin t\,dt\right]$. The problem is that the variable is the *lower* endpoint rather than the upper endpoint of the integral. To apply FTC I we need to switch the order of the limits. But that changes the sign of the integral (see Definition 1.5). So we have

$$\frac{d}{dx}\left[\int_x^0 t\sin t\,dt\right] = \frac{d}{dx}\left[-\int_0^x t\sin t\,dt\right] = -x\sin x,$$

because $t\sin t$ is continuous so FTC I applies.

There is also a chain rule version of the FTC I when the upper limit of integration is not just x but a function of x. Suppose for instance someone (me!) gave you the problem of determining

$$\frac{d}{dx}\left[\int_0^{x^3} \tan t\, dt\right].$$

We cannot apply FTC I directly because the upper limit of integration is x^3, not x. FTC I requires an x as the upper limit. However, we can make a substitution, like we sometimes do with the chain rule for derivatives. In general, if u is a function of x, then composition $y = A(u)$ is also really a function of x and so

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

In our case, let $u = x^3$ so $\frac{du}{dx} = 3x^2$. Now if $y = \int_0^{x^3} \tan t\, dt$, then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\int_0^{x^3} \tan t\, dt\right] \\
\frac{dy}{du}\cdot\frac{du}{dx} &= \frac{d}{du}\left[\int_0^{u} \tan t\, dt\right]\cdot\frac{du}{dx} && \text{let } u = x^3 \\
&= \tan(u)\cdot\frac{du}{dx} && \text{Use FTC with variable } u \\
&= \tan(x^3)\cdot 3x^2 && \text{Substitute back: } u = x^3 \text{ and } \tfrac{du}{dx} = 3x^2
\end{aligned}$$

So

$$\frac{d}{dx}\left[\int_0^{x^3} \tan t\, dt\right] = \tan(x^3)\cdot 3x^2.$$

Answer to YOU TRY IT 1.11 : The largest interval that contains 0 on which $\tan x^3$ is defined is $\left(\sqrt[3]{-\frac{\pi}{2}}, \sqrt[3]{\frac{\pi}{2}}\right)$.

YOU TRY IT 1.11. Ok, stop for a second. In Example 1.2.3, what interval must x lie in?

EXAMPLE 1.2.5. Put it all together: Determine $\frac{d}{dx}\left[\int_{e^{5x}}^0 \cos^2 t\, dt\right]$.

Solution. First we reverse the order of the endpoints so the the variable is the upper endpoint.

$$\frac{d}{dx}\left[\int_{e^{5x}}^0 \cos^2 t\, dt\right] = \frac{d}{dx}\left[-\int_0^{e^{5x}} \cos^2 t\, dt\right]$$

Now we use the chain rule with $u = e^{5x}$ to obtain

$$\begin{aligned}
\frac{d}{dx}\left[-\int_0^{e^{5x}} \cos^2 t\, dt\right] &= \frac{d}{du}\left[-\int_0^{u} \cos^2 t\, dt\right]\cdot\frac{du}{dx} && \text{let } u = e^{5x} \\
&= -\cos^2(u)\cdot\frac{du}{dx} && \text{Use FTC with variable } u \\
&= -\cos^2(e^{5x})\cdot 5e^{5x} && \text{Substitute back: } u = e^{5x} \text{ and } \tfrac{du}{dx} = 5e^{5x}
\end{aligned}$$

Not too bad!

Answer to YOU TRY IT 1.12 : $F'(x) = -(\sin x)e^{\cos x}$.

YOU TRY IT 1.12. Find the derivative of $F(x) = \int_\pi^{\cos x} e^t\, dt$.

YOU TRY IT 1.13. Here's one from your text. The graph of f is shown in Figure 1.42.

Define the following two accumulation or net area functions: $A(x) = \int_{-2}^x f(t)\, dt$ and $F(x) = \int_4^x f(t)\, dt$. Evaluate the following net area functions.

(a) $A(-2)$ (b) $F(8)$ (c) $A(4)$
(d) $F(4)$ (e) $A(8)$ (f) $F(-2)$

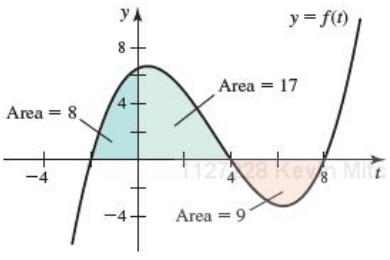

Figure 1.42: This is the graph of $f(t)$, for YOU TRY IT 1.13 .
Answer to YOU TRY IT 1.13 : (a) 0; (b) −9; (c) 25; (d) 0; (e) 16; (f) −25.

YOU TRY IT 1.14. OK, the Fundamental Theorem says that if we let $F(x) = \int_{-2}^{x} f(t)\,dt$, then $F'(x) = f(x)$. But also remember $F(x)$ is just the net area between f and the x axis on the interval from -2 to endpoint x. Answer the following questions.

(*g*) On what interval(s) is F decreasing? Explain.

(*h*) At what point(s), if any, does F have a local max? Min?

(*i*) Does F have any points of inflection? Explain.

(*j*) On what interval(s) is F concave down?

(*k*) Is $F(0)$ a positive number or negative? Explain.

YOU TRY IT 1.15. Suppose that $\int_1^x g(t)\,dt = x^2 \ln x$. Evaluate $g(2)$ and explain your answer. Hint: Apply FTC I.

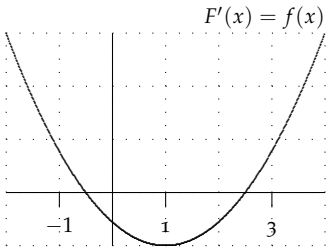

Figure 1.43: This is the graph of $F'(x) = f(x)$, not $F(x)$.

Answer to YOU TRY IT 1.14 : (a) $[-2, 1]$; (b) l. max: $x = -0.5$, l. min: $x = 1.5$; (c) $x = 1$; (d) $[-2, 1]$; (e) positive.

Answer to YOU TRY IT 1.15 : $g(2) = 4 \ln 2 + 2$.

1.3 *An Application of Definite Integrals: Average Value*

Here is another simple example of an application of the definite integral which points out the power of the definition of the integral as a Riemann sum.

Suppose that we want to know the average temperature for February 27, 2003 in Geneva (see Figure 1.44). How might we find it? Well, we could take the $n = 24$ hourly temperature recordings, add them together, and then divide by 24 might as we might do to find any average. Is the average 19.7 as listed in the table? What 'average' is that?

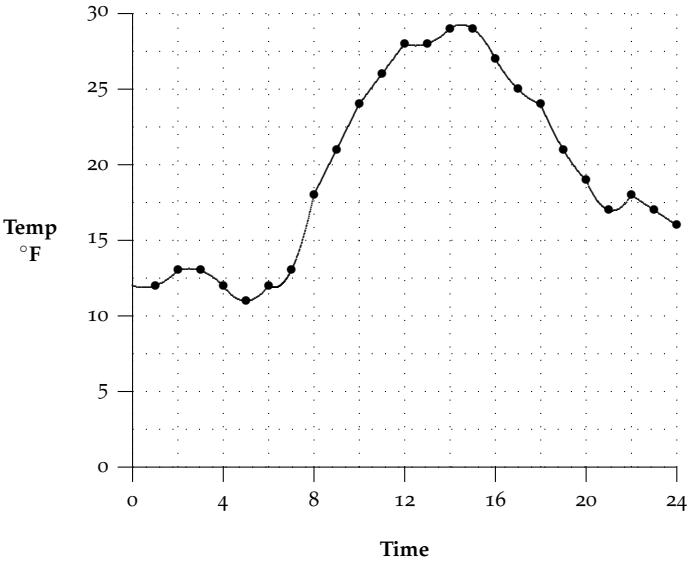

Figure 1.44: A graph of the temperature on February 27, 2003 using the data to the right.

Time	Temp
1:00	12
2:00	13
3:00	13
4:00	12
5:00	11
6:00	12
7:00	13
8:00	18
9:00	21
10:00	24
11:00	26
12:00	28
13:00	28
14:00	29
15:00	29
16:00	27
17:00	25
18:00	24
19:00	21
20:00	19
21:00	17
22:00	18
23:00	17
24:00	16
Ave	19.7

The average of 19.7 'privileges' those recordings made on the hour. We could get a better estimate if we recorded temperatures every half-hour, or every 5 minutes, or every minute, or perhaps every second. The more recordings we use, the better the 'average.' Let's generalize the problem. In doing so, we are sometimes able to see the pattern which will help us solve the particular problem we are interested in.

The Average Value Problem: Let f be a continuous function on the closed interval $[a, b]$. Find the average value of f on $[a, b]$.

SOLUTION. We make use of the outline of steps on page 22. But how do we subdivide an average and make it product? As usual, start by dividing $[a, b]$ into n equal subintervals with partition points $\{x_0, x_1, \ldots, x_n\}$. Then, as we suggested above,

$$\text{Average of } f \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \sum_{k=1}^{n} f(x_k) \cdot \frac{1}{n}. \tag{1.10}$$

The summation looks almost like a Riemann sum except we now have $\frac{1}{n}$ instead of Δx. But hold on!

$$\Delta x = \frac{b-a}{n}$$

so

$$\frac{1}{n} = \frac{b-a}{n} \cdot \frac{1}{b-a} = \frac{\Delta x}{b-a}.$$

Substituting this back in equation (1.10) gives

$$\text{Average of } f \approx \sum_{k=1}^{n} f(x_k) \cdot \frac{\Delta x}{b-a} = \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\Delta x. \tag{1.11}$$

Now we do have a Riemann sum in (1.11). We have already remarked that if we let n increase (take more points in our average), we should get a more accurate approximation. The best approximation occurs when we take a limit as the number of evaluation points $n \to \infty$. In other words

$$\text{Average of } f = \lim_{n\to\infty} \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.12}$$

We know this limit exists and equals the definite integral because f is continuous (see Theorem 1.3). Having carried out the steps on page 22, we are led to make the following definition.

DEFINITION 1.6 (Average Value)**.** Assume that f is integrable on $[a,b]$. Then the **average value of f on** $[a,b]$ is denoted by $\overline{f} = f_{\text{ave}}$ and is defined by

$$\overline{f} = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

.

EXAMPLE 1.3.1. Find the average value of $f(x) = \sqrt{x}$ on $[0,9]$.

SOLUTION. Using Definition 1.6

$$\overline{f} = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{9-0}\int_0^9 \sqrt{x}\,dx = \frac{1}{9}\cdot\frac{2}{3}x^{3/2}\Big|_0^3 = \frac{2}{27}(27-0) = 2.$$

The average value is shown in Figure 1.45. Think of the original curve as a wave in a fish tank. The wave settles to the average value of the function. The area of the rectangle formed using the average value as the height is the same as the area under the original curve. Notice that there is a point, namely $c = 4$ at which the height of the curve is the same as the average value ($f(4) = 2$).

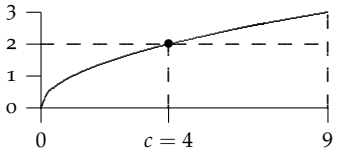

Figure 1.45: The average value of $\sqrt{x}$ on $[0,9]$ is 2. This value actually occurs at $c = 4$.

YOU TRY IT 1.16. Find the average value of $f(x) = x^2$ on $[-1,2]$. For which values c in the interval does the average value actually occur?

Answer to YOU TRY IT 1.16 : The average value is 1. It occurs at $c = \pm 1$.

EXAMPLE 1.3.2. A patient being treated for emphysema is tested with a spirometer to measure lung capacity. The data show the volume of air in the patient's lungs during inhalation is given by $V(t) = 1 - \cos\left(\frac{\pi t}{2}\right)$ pints over the time interval $[0,2]$ seconds. Find the average volume of air in the his lungs during this period.

SOLUTION. Using Definition 1.6

$$\begin{aligned}\text{Average V} &= \frac{1}{2-0}\int_0^2 1-\cos\left(\frac{\pi t}{2}\right)\,dt \\ &= \frac{1}{2}\left[t - \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right)\right]\Big|_0^2 = \frac{1}{2}[(2-0)-(0-0)] = 1 \text{ pint.}\end{aligned}$$

In Figure 1.46 notice how the area under the curve above the average value balances out the missing area below the average value and the curve. Notice also that the average value actually occurs at $c = 1$ (since $f(1) = 1$).

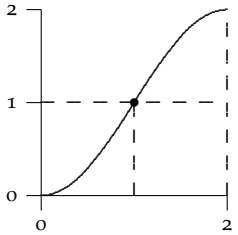

Figure 1.46: The average value of $V(t) = 1 - \cos\left(\frac{\pi t}{2}\right)$ on $[0,2]$ is 1.

YOU TRY IT 1.17. A patient being treated for pulmonary fibrosis is tested with a spirometer to measure lung capacity. The data show the volume of air in the patient's lungs during both the inhalation and exhalation cycles is given by

$$V(t) = 1 - \cos\left(\frac{2\pi t}{5}\right) \text{ pints}$$

over the time interval $[0,5]$ seconds. Find the average volume of air in the his lungs during this period. At what time(s) does this volume occur?

Answer to YOU TRY IT 1.17 : 1 pint and it occurs at $c = \frac{5}{4}$ and $\frac{15}{4}$.

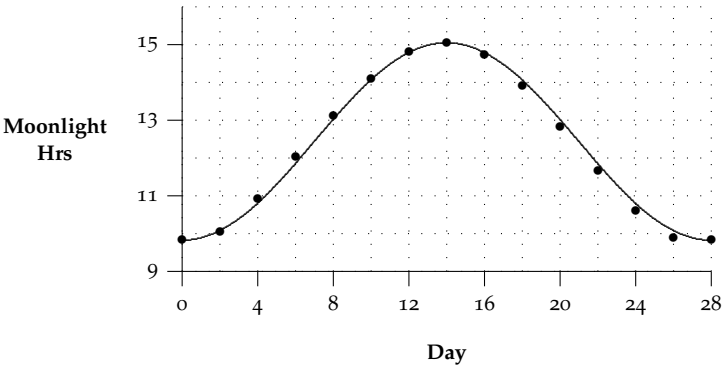

Figure 1.47: Data (for Boston) from *The Old Farmer's Almanac* for the number of hours the moon is visible per day during its 28-day cycle.

YOU TRY IT 1.18 (Hand in: Challenge). The phases of the moon occur in a regular, predictable 28 day lunar cycle. Data (for Boston) from *The Old Farmer's Almanac* indicate that the hours, $v(x)$, the moon is visible per day over the course of the cycle is $v(x) = 2.615\cos(\frac{\pi}{14}x - \pi) + 12.435$, where x is the day of the cycle.

(*a*) Find the average number of hours the moon is visible per day in Boston using calculus

(*b*) Extra Credit: Use your brain to explain why you should have expected this answer!

In Example 1.3.1 and Example 1.3.2 we noted that there were points where the average value of the function actually occurred. This turns out to always be true as long as $f(x)$ is continuous.

THEOREM 1.10 (Mean Value Theorem for Integrals: **MVTI**). If f is continuous on $[a, b]$, then there's a point c in $[a, b]$ so that

$$\int_a^b f(t)\,dt = f(c)\cdot(b-a).$$

In other words

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt = \overline{f} = f_{\text{ave}}.$$

Proof. As usual, for x in $[a, b]$, define our accumulation function as

$$A(x) = \int_a^x f(t)\,dt.$$

Then by FTC I, $A'(x) = f(x)$. So $A(x)$ is differentiable on $[a, b]$ so it is continuous there. By the original MVT, there is a point c in $[a, b]$ so that

$$A'(c) = \frac{A(b)-A(a)}{b-a} = \frac{\int_a^b f(t)\,dt - \int_a^a f(t)\,dt}{b-a} = \frac{\int_a^b f(t)\,dt - 0}{b-a} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

But $A'(c) = f(c)$, so

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt = \overline{f} = f_{\text{ave}}.$$

□

EXAMPLE 1.3.3. Let $f(x) = 2\cos(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find the point c in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where $f(c) = f_{ave}$.

SOLUTION. First notice that $2\cos x$ is differentiable so it is continuous. So the MVTI applies. Next we need to determine $\overline{f} = f_{\text{ave}}$. Using Definition 1.6

$$\overline{f} = f_{\text{ave}} = \frac{1}{\pi/2-(-\pi/2)}\int_{-\pi/2}^{\pi/2} 2\cos x\,dx = \frac{1}{\pi}(2\sin x)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}[2-(-2)] = \frac{4}{\pi}.$$

So we need to find c in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $f(c) = \overline{f} = f_{\text{ave}} = \frac{4}{\pi}$. In other words,

$$\begin{aligned} f(c) = 2\cos c &= \frac{4}{\pi} \\ \cos c &= \frac{2}{\pi} \\ c &= \arccos\left(\frac{2}{\pi}\right) \approx \pm 0.8806892354. \end{aligned}$$

YOU TRY IT 1.19. Let $f(x) = x^2 - 1$ on $[0,3]$. Does the MVTI apply to this function? Why? If so, find the point(s) c in $[0,3]$ so that $f(c) = \overline{f} = f_{\text{ave}}$.

Answer to YOU TRY IT 1.19 : The MVTI does apply because f is a polynomial (continuous). $c = \sqrt{3}$.

EXAMPLE 1.3.4. Find the average value of $f(x) = |x-3|$ on $[0,5]$ and determine the points c where the average occurs.

SOLUTION. By Definition 1.6,

$$\begin{aligned} \overline{f} = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{5-0}\int_0^5 |x-3|\,dx \\ \overset{\text{See graph}}{=} \frac{1}{5}\left[\frac{1}{2}(3)(3) + \frac{1}{2}(2)(2)\right] \\ = 1.3. \end{aligned}$$

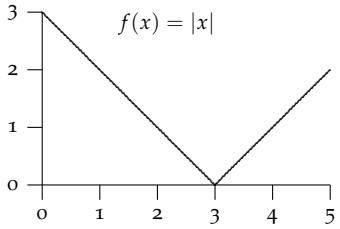

Figure 1.48: The area under $f(x) = |x-3|$ on $[0,5]$ consists of triangles.

Note that we were able to easily evaluate the integral by using the geometry of the the two triangles.

Since $|x-3|$ is continuous, the MVTI says there's a point c in $[0,5]$ where $f(c) = \overline{f} = f_{\text{ave}}$. We need

$$|c-3| = 1.3 \iff \begin{cases} c-3 = 1.3 \\ c-3 = -1.3 \end{cases} \iff \begin{cases} c = 4.3 \\ c = 1.7 \end{cases}.$$

Both points are in the interval.