Motion: Velocity and Net Change

In Calculus I you interpreted the first and second derivatives as velocity and acceleration in the context of motion. So let's apply the initial value problem results to motion problems. Recall that

- $s(t) =$ position at time t.
- $s'(t) = v(t) =$ velocity at time t.
- $s''(t) = v'(t) = a(t) =$ acceleration at time t.

Therefore

- $\int a(t)\,dt = v(t) + c_1 =$ velocity.
- $\int v(t)\,dt = s(t) + c_2 =$ position at time t.

We will need to use additional information to evaluate the constants c_1 and c_2.

EXAMPLE 6.1. Suppose that the acceleration of an object is given by $a(t) = 2 - \cos t$ for $t \geq 0$ with

- $v(0) = 1$, this is also denoted v_0
- $s(0) = 3$, this is also denoted s_0.

Find $s(t)$.

SOLUTION. First find $v(t)$ which is the antiderivative of $a(t)$.

$$v(t) = \int a(t)\,dt = \int 2 - \cos t\,dt = 2t - \sin t + c_1.$$

Now use the initial value for $v(t)$ to solve for c_1:

$$v(0) = 0 - 0 + c_1 = 1 \Rightarrow c_1 = 1.$$

Therefore, $v(t) = 2t - \sin t + 1$. Now solve for $s(t)$ by taking the antiderivative of $v(t)$.

$$s(t) = \int v(t)\,dt = \int 2t - \sin t + 1\,dt = t^2 + \cos t + t + c_2$$

Now use the initial value of s to solve for c_2:

$$s(0) = 0 + \cos 0 + c_2 = 3 \Rightarrow 1 + c_2 = 3 \Rightarrow c_2 = 2.$$

So $s(t) = t^2 + \cos t + 2t + 2$.

EXAMPLE 6.2. If acceleration is given by $a(t) = 10 + 3t - 3t^2$, find the exact position function if $s(0) = 1$ and $s(2) = 11$.

SOLUTION. First

$$v(t) = \int a(t)\,dt = \int 10 + 3t - 3t^2\,dt = 10t + \tfrac{3}{2}t^2 - t^3 + c.$$

Now

$$s(t) = \int 10t + \tfrac{3}{2}t^2 - t^3 + c\,dt = 5t^2 + \tfrac{1}{2}t^3 - \tfrac{1}{4}t^4 + ct + d.$$

But $s(0) = 0 + 0 - 0 + 0 + d = 1$ so $d = 1$. Then $s(2) = 20 + 4 - 4 + 2c + 1 = 11$ so $2c = -10 \Rightarrow c = -5$. Thus, $s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 - 5t + 1$.

EXAMPLE 6.3. If acceleration is given by $a(t) = \sin t + \cos t$, find the position function if $s(0) = 1$ and $s(2\pi) = -1$.

SOLUTION. First

$$v(t) = \int a(t)\,dt = \int \sin t + \cos t\,dt = -\cos t + \sin t + c.$$

Now

$$s(t) = \int -\cos t + \sin t + c\,dt = -\sin t - \cos t + ct + d.$$

But $s(0) = 0 - 1 + 0 + 0 + d = 1$ so $d = 2$. Then $s(2\pi) = 0 - 1 + 2c\pi + 2 = -1$ so $2\pi c = -2 \Rightarrow c = -\frac{1}{\pi} + 2$. Thus, $s(t) = \cos t + \sin t - \frac{1}{\pi}t$.

Displacement vs Distance Travelled

DEFINITION 6.1. The **displacement** of an object between times $t = a$ and a later time $t = b$ is

$$s(b) - s(a) = \int_a^b v(t)\,dt.$$

The **distance travelled** by an object between times $t = a$ and a later time $t = b$ is

$$\text{Distance travelled} = \int_a^b |v(t)|\,dt.$$

In other words, **displacement** is the **net area** between the velocity curve and the horizontal axis, while the **distance travelled** is the **total area** between the velocity curve and the horizontal axis.

EXAMPLE 6.4. Suppose an object moves with velocity $2t^2 - 12t + 16$ km/hr.

(1) Determine the displacement of the object on the time interval $[1,3]$ and $[0,4]$ and interpret your answer.

(2) Determine the distance travelled on $[0,4]$

SOLUTION. (1) The displacement is easy to calculate: For the interval $[0,4]$, On $[0,4]$,

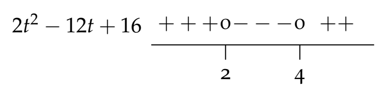
$$s(4) - s(0) = \int_0^4 2t^2 - 12t + 16\,dt = \frac{2t^3}{3} - 6t^2 + 16t\Big|_0^4 = \frac{128}{3} - 96 + 64 - 0 = \frac{32}{3}.$$

On $[1,3]$,

$$s(3) - s(1) = \int_1^3 2t^2 - 12t + 16\,dt = \frac{2t^3}{3} - 6t^2 + 16t\Big|_1^4 = (18 - 54 + 48) - \left(\frac{2}{3} - 6 + 16\right) = \frac{4}{3}.$$

(2) The distance travelled is harder to determine since we need to integrate $|v(t)|$. We must first determine where $v(t)$ is positive and negative.

$$2t^2 - 12t + 16 = 2(t^2 - 6t + 8) = 2(t-2)(t-4) = 0 \Rightarrow t = 2, 4.$$



The number line to the right shows that $2t^2 - 12t + 16 \leq 0$ only $[2,4]$. We can now find the distance travelled (total area) by splitting the interval into two pieces $[0,2]$ and $[2,4]$, changing the sign of $v(t)$ on the second piece to obtain the absolute value of $v(t)$.

$$\begin{aligned}
\text{Dist Trav} = \text{Total Area} &= \int_0^4 |2t^2 - 12t + 16|\,dt \\
&= \int_0^2 2t^2 - 12t + 16\,dt + \int_2^4 -(2t^2 - 12t + 16)\,dt \\
&= -\left[\frac{2t^3}{3} - 6t^2 + 16t\Big|_0^2\right] - \left[\frac{2t^3}{3} - 6t^2 + 16t\Big|_2^4\right] \\
&= \frac{40}{3} + \frac{8}{3} = 16.
\end{aligned}$$

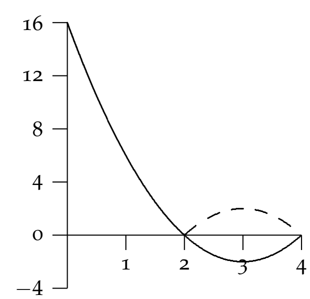


Figure 6.1: The distance travelled on $[0,4]$ is the area under the absolute value of the velocity curve.

EXAMPLE 6.5. Suppose an object moves with velocity $t^3 - 5t^2 + 4t$ m/s.

(1) Determine the displacement of the object on the time interval $[0,6]$ and interpret your answer.

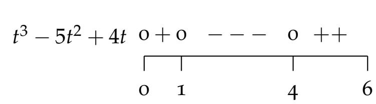
(2) Determine the distance travelled on $[0,6]$

SOLUTION. (1) For displacement on $[0,6]$,

$$\int_0^6 t^3 - 5t^2 + 4t\,dt = \frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\Big|_0^6 = (324 - 360 + 72) - 0 = 36.$$

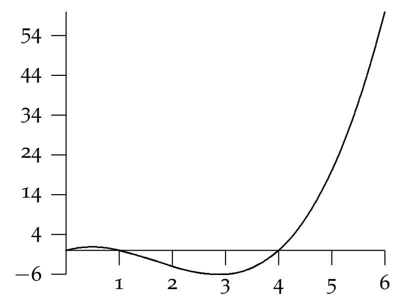
(2) For the distance travelled we must first determine where $v(t)$ is positive and negative.

$$t^3 - 5t^2 + 4t = t(t^2 - 5t + 4) = t(t-1)(t-4) = 0 \Rightarrow t = 0, 1, 4.$$



The number line to the right shows that $t^3 - 5t^2 + 4t \leq 0$ only $[1, 4]$. We can now find the distance travelled (total area) by splitting the interval into three pieces $[0, 1]$, $[1, 4]$ and $[4, 6]$, changing the sign of $v(t)$ on the second piece to obtain the absolute value of $v(t)$.

$$\begin{aligned}\text{Dist Trav} &= \int_0^6 |2t^2 - 12t + 16|\,dt \\ &= \int_0^1 t^3 - 5t^2 + 4t\,dt - \int_1^4 t^3 - 5t^2 + 4t\,dt + \int_4^6 t^3 - 5t^2 + 4t\,dt \\ &= \left[\frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\Big|_0^1\right] - \left[\frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\Big|_1^4\right] + \left[\frac{t^4}{4} - \frac{5t^3}{3} + 2t^2\Big|_4^6\right] \\ &= \frac{7}{12} + \frac{45}{4} + \frac{140}{3} = \frac{117}{2}.\end{aligned}$$



Constant Acceleration: Gravity

In many motion problems the acceleration is constant. This happens when an object is thrown or dropped and the only acceleration is due to gravity. In such a situation we have

- $a(t) = a$, constant acceleration
- with initial velocity $v(0) = v_0$
- and initial position $s(0) = s_0$.

Then

$$v(t) = \int a(t)\,dt = \int a\,dt = at + c.$$

But

$$v(0) = a \cdot 0 + c = v_0 \Rightarrow c = v_0.$$

So

$$\boxed{v(t) = at + v_0.}$$

Next,

$$s(t) = \int v(t)\,dt = \int at + v_0\,dt = \tfrac{1}{2}at^2 + v_0 t + c.$$

At time $t = 0$,

$$s(0) = \tfrac{1}{2}a(0)^2 + v_0(0) + c = s_0 \Rightarrow c = s_0.$$

Therefore

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0.}$$

EXAMPLE 6.6. Suppose a ball is thrown with initial velocity 96 ft/s from a roof top 432 feet high. The acceleration due to gravity is constant $a(t) = -32$ ft/s^2. Find $v(t)$ and $s(t)$. Then find the maximum height of the ball and the time when the ball hits the ground.

SOLUTION. Recognizing that $v_0 = 96$ and $s_0 = 432$ and that the acceleration is constant, we may use the general formulas we just developed.

$$\boxed{v(t) = at + v_0 = -32t + 96}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + 96t + 432.}$$

The max height occurs when the velocity is 0 (when the ball stops rising):

$$v(t) = -32t + 96 = 0 \Rightarrow t = 3 \Rightarrow s(3) = -144 + 288 + 432 = 576 \text{ ft.}$$

The ball hits the ground when $s(t) = 0$.

$$s(t) = -16t^2 + 96t + 432 = -16(t^2 - 6t - 27) = -16(t-9)(t+3) = 0.$$

So $t = 9$ only (since $t = -3$ does not make sense).

EXAMPLE 6.7. A person drops a stone from a bridge. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

SOLUTION. Here's what we know. $v_0 = 0$ (dropped) and $s(5) = 0$ (hits water). And we know acceleration is constant, $a = -32$ ft/s^2. We want to find the height of the bridge, which is just s_0. Use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = -32t}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + s_0.}$$

Now we use the position we know: $s(5) = 0$.

$$s(5) = -16(5)^2 + s_0 \Rightarrow s_0 = 400 \text{ ft.}$$

Notice that we did not need to use the velocity function.

YOU TRY IT 6.1 (Extra Credit). In the previous problem we did not take into account that sound does not travel instantaneously in your calculation above. Assume that sound travels at 1120 ft/s. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it?

Check on your answer: Should the bridge be higher or lower than in the preceding example? Why?

EXAMPLE 6.8. Here's a variation. This time we will use metric units. Suppose a ball is thrown with unknown initial velocity v_0 m/s from a roof top 49 meters high and the position of the ball at time $t = 3$ is $s(3) = 0$. The acceleration due to gravity is constant $a(t) = -9.8$ m/s^2. Find $v(t)$ and $s(t)$.

SOLUTION. This time v_0 is unknown but $s_0 = 49$ and $s(3) = 0$. Again the acceleration is constant so we may use the general formulas for this situation.

$$v(t) = at + v_0 = -9.8t + v_0$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -4.9t^2 + v_0 t + 49.$$

But we know that

$$s(3) = -4.9(3)^2 + v_0 \cdot 3 + 49 = 0$$

which means

$$3v_0 = 4.9(9) - 4.9(10) = -4.9 \Rightarrow v_0 = -4.9/3.$$

So

$$\boxed{v(t) = -9.8t - \tfrac{49}{30}}$$

and

$$\boxed{s(t) = -4.9t^2 - \tfrac{49}{30}t + 49.}$$

Interpret $v_0 = -4.9/3$.

EXAMPLE 6.9. Mo Green is attempting to run the 100m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his *acceleration* is constant, a, over the entire race. Determine his velocity function. (a will still appear as an unknown constant.) Determine his position function. There should be no unknown constants in your equation at this point. What is his velocity at the end of the race? Do you think this is realistic?

SOLUTION. We have: constant acceleration $= a\,\text{m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2} = 2.0825\,\text{m/s}^2$. So $s(t) = 2.0825t^2$. Mo's velocity at the end of the race is $v(9.8) = a \cdot 9.8 = 2.0825(9.8) = 20.41$ m/s...not realistic.

EXAMPLE 6.10. A stone dropped off a cliff hits the ground with speed of 120 ft/s. What was the height of the cliff?

SOLUTION. Notice that $v_0 = 0$ (dropped!) and s_0 is unknown but is equal to the cliff height, and that the acceleration is constant $a = -32$ ft/. Use the general formulas for motion with constant acceleration:

$$\boxed{v(t) = at + v_0 = -32t + 0 = -32t}.$$

Now we use the velocity function and the one velocity value we know: $v = -120$ when it hits the ground. So the *time* when it hits the ground is given by

$$v(t) = -32t = -120 \Rightarrow t = 120/32 = 15/4$$

when it hits the ground. Now remember when it hits the ground the height is 0. So $s(15/4) = 0$. But we know

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -16t^2 + 0t + s_0 = -16t^2 + s_0.}$$

Now substitute in $t = 15/4$ and solve for s_0.

$$s(15/4) = 0 \Rightarrow -16(15/4)^2 + s_0 = 0 \Rightarrow s_0 = 15^2 = 225.$$

The cliff height is 225 feet.

EXAMPLE 6.11. A car is traveling at 90 km/h when the driver sees a deer 75 m ahead and slams on the brakes. What constant deceleration is required to avoid hitting Bambi? [Note: First convert 90 km/h to m/s.]

SOLUTION. Let's list all that we know. $v_0 = 90$ km/h or $\frac{90000}{60 \cdot 60} = 25$ m/s and $s_0 = 0$. Let time t^* represent the time it takes to stop. Then $s(t^*) = 75$ m. Now the car is stopped at time t^*, so we know $v(t^*) = 0$. Finally we know that acceleration is an unknown constant, a, which is what we want to find.

Now we use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = at + 25}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2 + 25t.}$$

Now use the other velocity and position we know: $v(t^*) = 0$ and $s(t^*) = 75$ when the car stops. So

$$v(t^*) = at^* + 25 = 0 \Rightarrow t^* = -25/a$$

and

$$s(t^*) = \tfrac{1}{2}a(t^*)^2 + 25t^* = \tfrac{1}{2}a(-25/a)^2 + 25(-25/a) = 75.$$

Simplify to get

$$\frac{625a}{2a^2} - \frac{625}{a} = \frac{625}{2a} - \frac{1350}{2a} = -\frac{625}{2a} = 75 \Rightarrow 150a = -625$$

so

$$a = -\frac{625}{150} = -\frac{25}{6}\ \text{m/s}.$$

(Why is acceleration negative?)

EXAMPLE 6.12. One car intends to pass another on a back road. What constant acceleration is required to increase the speed of a car from 30 mph (44 ft/s) to 50 mph ($\frac{220}{3}$ ft/s) in 5 seconds?

SOLUTION. Given: $a(t) = a$ constant. $v_0 = 44$ ft/s. $s_0 = 0$. And $v(5) = \frac{220}{3}$ ft/s. Find a. But

$$v(t) = at + v_0 = at + 44.$$

So

$$v(5) = 5a + 44 = \frac{220}{3} \Rightarrow 5a = \frac{220}{3} - 44 = \frac{88}{3}.$$

Thus $a = \frac{88}{15}$.

YOU TRY IT 6.2. A toy bumper car is moving back and forth along a straight track. Its acceleration is $a(t) = \cos t + \sin t$. Find the particular velocity and position functions given that $v(\pi/4) = 0$ and $s(\pi) = 1$.

Answer to YOU TRY IT 6.2 : $v(t) = \int a(t)\,dt = \int \cos t + \sin t\,dt = \sin t - \cos t + c$. So $v(\pi/4) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + c = 0 \Rightarrow c = 0$. Thus, $v(t) = \sin t - \cos t$. Now $s(t) = \int v(t)\,dt = \int \sin t - \cos t\,dt = -\cos t - \sin t + c$. Since $s(\pi) = -(-1) - 0 + c = 1 \Rightarrow c = 0$. So $s(t) = -\cos t - \sin t$.