Rational Functions and Partial Fractions

Our final integration technique deals with the class of functions known as rational functions. Recall from Calculus I that

DEFINITION 7.1. A **rational function**[1] is a function that is the ratio of two polynomials

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials. (Remember a *polynomial* has the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_i are real constants and n is a non-negative integer and is called the *degree* of the polynomial.)

[1] Here 'rational' means 'ratio', as in 'the ratio of two polynomials.'

Here are several examples of rational functions; identify the polynomials $p(x)$ and $q(x)$.

- $r(x) = \dfrac{x^2+7}{3x^5+7x}$
- $s(x) = \dfrac{2x^3-9x+1}{x}$
- $t(x) = \dfrac{1}{x^2+1}$
- $r(x) = 3x^{-2} + 2x + 1$ is also rational because it can be written as a ratio of two polynomials
$$r(x) = \frac{3}{x^2} + 2x + 1 = \frac{2x^3+x^2+3}{x^2}.$$

A couple of non-examples include

- $z(x) = \dfrac{x^{3/2}+7}{3x+7x}$ because the term $x^{3/2}$ is not a polynomial since the power is not a non-negative integer.
- $s(x) = \dfrac{\sin(x^2+1)}{x}$ is not a polynomial since it contains a trigonometric function.

Remember from Calculus I that rational functions are continuous and differentiable at all points in their domains, i.e., at all points where the denominator is not 0.

Our main concern in this chapter is to determine when such rational functions can be integrated. From our earlier work, we know that any interval on which a rational function is continuous it is also integrable. But, how do we actually find an antiderivative for the function? For instance, to check that you can integrate

$\int \frac{1}{1+x^2}\,dx$, $\int \frac{x}{1+x^2}\,dx$, $\int \frac{3x^2-1}{x^3-x}\,dx$, and $\int \frac{1}{1+x}\,dx$. But what about very similar looking functions such as $\int \frac{2x+1}{x^3-x}\,dx$ or $\int \frac{4}{4-x^2}\,dx$? Why are these integrals not so easy to do? Over the next few sections, we examine a series of special cases of rational functions that we will see are relatively easy to integrate using a technique known as *partial fractions*.

7.1 Special Cases: Linear Factors with Degree $p(x) < q(x)$

There are several techniques that can be used integrate rational functions. We will concentrate on a single technique and a couple of simple variations that work with rational functions of a particular type.

Assume we have a rational function of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where degree of $p(x) <$ degree of $q(x)$ and $q(x)$ can be factored into *linear factors* (factors of degree 1). Some examples include

- $\frac{2x+1}{x^3-x} = \frac{2x+1}{x(x^2-1)} = \frac{2x+1}{x(x-1)(x+1)}$ where all the factors in the denominator are linear and different and degree $p(x) = 1$ and degree $q(x) = 3$.
- $\frac{4}{4-x^2} = \frac{4}{(2-x)(2+x)}$; the two factors in the denominator are linear and different and degree of $p(x) <$ degree of $q(x)$.
- $\frac{6x-2}{x^2-2x-3} = \frac{6x-2}{(x-3)(x+1)}$ where all the factors in the denominator are linear and different and degree of $p(x) <$ degree of $q(x)$.
- $\frac{x+4}{x^3+x^2} = \frac{x+4}{x \cdot x(x+1)}$; the three factors in the denominator are linear and degree of $p(x) <$ degree of $q(x)$.

On the other hand the rational function

$$\frac{2x+1}{x^3+x} = \frac{2x+1}{x(x^2+1)}$$

does not satisfy our criterion above because the denominator contains a factor of degree 2, in particular, x^2+1 cannot be written as a product of linear factors.

We will use a technique called partial fractions to integrate the special rational functions above.[2]

[2] Partial fractions can be used to integrate other types of rational functions. Your text has further examples.

The Key to Partial Fractions

Let's look at a couple of ordinary fractions and how they can be rewritten in 'simpler' terms. Notice

$$\frac{1}{20} = \frac{1}{4} - \frac{1}{5}$$

or

$$\frac{10}{21} = \frac{1}{3} + \frac{1}{7}.$$

In each case the original fraction has been rewritten in terms of simpler component fractions. The idea is to do the same for rational functions. For example, can we write

$$\frac{4}{4-x^2} \quad \text{as} \quad \frac{A}{2-x}+\frac{B}{2+x}$$

where $2-x$ and $2+x$ are the linear factors of $4-x^2$? If we could, then putting the two simpler pieces back over a common denominator would give us

$$\frac{4}{4-x^2}=\frac{A}{2-x}+\frac{B}{2+x}=\frac{2A+Ax+2B-Bx}{4-x^2}=\frac{(A-B)x+(2A+2B)}{4-x^2}. \tag{7.1}$$

The first and last rational functions in (7.1) are equal and have the same denominator. The only way that this can happen is if the numerators are also the same. So (7.1) means

$$4=(A-B)x+(2A+2B). \tag{7.2}$$

But (7.2) is true if and only if the x-terms on each side are equal and the constants on each side are equal. Since there are no x's on the left side and $A-B$ on the right, we must have

$$x\text{'s}: \quad 0=A-B. \tag{7.3}$$

Comparing the constant terms in the same way we have

$$\text{constants}: \quad 4=2A+2B. \tag{7.4}$$

From (7.3) we see that $A=B$ and using this in (7.4) gives

$$4=4A.$$

Thus, $A=1$. Putting $A=1$ in (7.3) or (7.4) makes $B=1$. So we see that

$$\frac{4}{4-x^2}=\frac{1}{2-x}+\frac{1}{2+x}. \tag{7.5}$$

Check that this is correct! We describe this process by saying that we have rewritten $\frac{4}{4-x^2}$ as $\frac{1}{2-x}+\frac{1}{2+x}$ using *partial fractions*.

So what! How do we use this? Well, suppose we need to solve $\int \frac{4}{4-x^2}\,dx$. Using (7.5) we can rewrite it and then integrate (using a 'mental adjustment'):

$$\begin{aligned}\int \frac{4}{4-x^2}\,dx &= \int\left(\frac{1}{2-x}+\frac{1}{2+x}\right)dx\\ &= -\ln|2-x|+\ln|2+x|+c=\ln\left|\frac{2-x}{2+x}\right|+c.\end{aligned}$$

7.2 *The Easiest Case: Distinct Linear Factors*

We can always carry out the same sort of process as above under the following circumstances:[3]

> *The Easiest Case:* Let $r(x)=\dfrac{r(x)}{q(x)}$ be a rational function. Assume that the denominator $q(x)$ of the rational function factors into *distinct* linear and the degree of the numerator $p(x)$ is less than the degree of the denominator $q(x)$. Then $r(x)=\dfrac{p(x)}{q(x)}$ can be rewritten using partial fractions.

[3] This is not so easy to prove in general; usually one sees the proof of this result in a graduate-level abstract algebra course.

EXAMPLE 7.2.1 (Partial Fractions: Easiest Case). Here's another example that uses **partial fractions**. Determine

$$\int \frac{3}{x^2+3x+2}\,dx.$$

SOLUTION. This is not an integral that we can immediately do, even with integration by parts. [4] So we try partial fractions. Notice that the degree of the numerator is less than the degree of the denominator ($0 < 2$) and the denominator factors into distinct linear factors: $(x+1)(x+2)$. We form the partial fractions $\frac{A}{x+1}$ and $\frac{B}{x+2}$, where A and B are constants. We solve for A and B as we did in the previous example.

[4] Don't confuse the terms *partial fractions* with *integration by parts.*

$$\frac{3}{x^2+3x+2} = \frac{3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

By putting the last terms together again, we get

$$\frac{3}{x^2+3x+2} = \frac{Ax+2A+Bx+B}{x^2+3x+2} = \frac{(A+B)x+(2A+B)}{x^2+3x+2}.$$

Since the denominators are the same, the numerators must be the same, too. In particular there are as many x's on the left side as on the right. There are none on the left and $A+B$ on the right side. Similarly for the constants.

$$\begin{array}{lrcl} x\text{'s:} & 0 & = & A+B \\ \text{constants:} & 3 & = & 2A+B \end{array}$$

Subtracting the first equation from the second gives

$$3 = A.$$

Substituting $A = 3$ into first or second equation makes $B = -3$. So we see that

$$\frac{3}{x^2+3x+2} = \frac{3}{x+1} - \frac{3}{x+2}.$$

Now on to integration:

$$\begin{aligned} \int \frac{3}{x^2+3x+2}\,dx &= \int \left(\frac{3}{x+1} - \frac{3}{x+2}\right)\,dx \\ &= 3\ln|x+1| - 3\ln|x+2| + c = 3\ln\left|\frac{x+1}{x+2}\right| + c. \end{aligned}$$

EXAMPLE 7.2.2 (Partial Fractions: Easiest Case). Determine

$$\int \frac{6x-2}{x^2-2x-3}\,dx.$$

SOLUTION. This is not an integral that we can immediately do with substitution or integration by parts. So we try partial fractions. The degree of the numerator is less than the degree of the denominator ($1 < 2$) and the denominator factors into distinct linear factors: $(x-3)(x+1)$. We form the partial fractions $\frac{A}{x-3}$ and $\frac{B}{x+1}$, where A and B are constants.

$$\frac{6x-2}{x^2-2x-3} = \frac{6x-2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{Ax+A+Bx-3B}{x^2-2x-3}.$$

So comparing the numerators of the first and last functions

$$\begin{array}{lrcl} x\text{'s:} & 6 & = & A+B \\ \text{constants:} & -2 & = & A-3B \end{array}$$

Subtracting the second equation from the first gives

$$8 = 4B.$$

So $B = 2$. Using this in the first or second equation makes $A = 4$. Now on to integration (using an adjustment):

$$\int \frac{6x-2}{x^2-2x-3}\,dx = \int \left(\frac{4}{x-3} + \frac{2}{x+1}\right)\,dx = 4\ln|x-3| + 2\ln|x+1| + c.$$

YOU TRY IT 7.1. Does it matter which letters you put over each linear factor? What would the numerator in the original integral have to be to make the problem a substitution problem?

Answers to YOU TRY IT 7.1 : No. It would have to be a multiple of $2x-2$.

EXAMPLE 7.2.3 (Partial Fractions: Easiest Case). Here's another quick example. Determine

$$\int \frac{2x+5}{x^2+2x-8}\,dx.$$

SOLUTION. Notice this is not quite a substitution integral, but partial fractions will work. The degree of the numerator is less than the degree of the denominator $(1 < 2)$ and the denominator factors into distinct linear factors: $(x-2)(x+4)$.

$$\frac{2x+5}{x^2+2x-8} = \frac{2x+5}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4} = \frac{Ax+4A+Bx-2B}{x^2+2x-8}.$$

Comparing the numerators of the first and last functions and solving for A and B gives

$$\begin{array}{lrclcrcl}
x\text{'s:} & 2 & = & A+B & \Rightarrow & 4 & = & 2A+2B \\
\text{constants:} & 5 & = & 4A-2B & & 5 & = & 4A-2B \\
\hline
 & & & & & 9 & = & 6A
\end{array}$$

So $A = \frac{3}{2}$ and $B = \frac{1}{2}$. The integration becomes:

$$\int \frac{2x+5}{x^2+2x-8}\,dx = \int \left(\frac{\frac{3}{2}}{x-2} + \frac{\frac{1}{2}}{x+4} \right) dx = \tfrac{3}{2}\ln|x-2| + \tfrac{1}{2}\ln|x+4| + c.$$

YOU TRY IT 7.2. What would the numerator in the original integral have to be to make the problem a substitution problem?

7.3 A Complication: Higher Degrees

The first complication that arises is is that the denominator of the rational function may factor into three or more distinct linear factors. The solution method works the same way as above, but it may be more complicated to find the constants.

EXAMPLE 7.3.1 (Partial Fractions: Three Distinct Linear Factors). Determine

$$\int \frac{2x^2-6x+2}{x^3-3x^2+2x}\,dx.$$

SOLUTION. Check that this is not quite a substitution integral. However, the degree of the numerator is less than the degree of the denominator $(2 < 3)$ and the denominator factors into distinct three linear factors: $x(x^2-3x+2) = x(x-1)(x-2)$. We form the partial fractions with a different constant for each linear factor:

$$\begin{aligned}
\frac{2x^2-6x+2}{x^3-3x^2+2x} = \frac{2x^2-6x+2}{x(x-1)(x-2)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \\
&= \frac{A(x-1)(x-2) + Bx(x-2) + Cx(x-1)}{x^3-3x^2+2x} \\
&= \frac{Ax^2-3Ax+2A+Bx^2-2B+Cx^2-Cx}{x^3-3x^2+2x}.
\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{lrcllrcl}
x^2\text{'s:} & 2 & = & A+B+C & \Rightarrow & 1 & = & B+C \\
x\text{'s:} & -6 & = & -3A-2B-C & \Rightarrow & -3 & = & -2B-C \\
\hline
\text{const:} & 2 & = & 2A \quad \Rightarrow A=1 & & -2 & = & -B
\end{array}$$

Notice that we used the value $A = 1$ to simplify the first two equations. It follows that $B = 2$ and $C = -1$. The integration becomes:

$$\int \frac{2x^2-6x+2}{x^3-3x^2+2x}\,dx = \int \frac{1}{x} + \frac{2}{x-1} - \frac{1}{x-2}\,dx = \ln|x| + 2\ln|x-1| - \ln|x-2| + c.$$

YOU TRY IT 7.3. What would the numerator in the original integral have to be to make the problem a substitution problem?

EXAMPLE 7.3.2 (Partial Fractions: Three Distinct Linear Factors). Determine

$$\int \frac{x^2+4x-1}{x^3-x}\,dx.$$

SOLUTION. Check that this is not a substitution integral. However, the degree of the numerator is less than the degree of the denominator $(2 < 3)$ and the denominator factors into distinct three linear factors: $x(x^2-1) = x(x-1)(x+1)$. We form the partial fractions with a different constant for each linear factor:

$$\begin{aligned}\frac{x^2+4x-1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} &= \frac{A(x^2-1)+Bx(x+1)+Cx(x-1)}{x^3-x}\\ &= \frac{Ax^2-A+Bx^2+B+Cx^2-Cx}{x^3-x}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{lrcrlrcl} x^2\text{'s:} & 1 & = & A+B+C & & \Rightarrow\ 0 & = & B+C \\ x\text{'s:} & 4 & = & B-C & & \Rightarrow\ 4 & = & B-C \\ \hline \text{const:} & -1 & = & -A & \Rightarrow A=1 & 4 & = & 2B \end{array}$$

Notice that we used the value $A = 1$ to simplify the first two equations. It follows that $B = 2$ and $C = -2$. The integration becomes:

$$\begin{aligned}\int \frac{x^2+4x-1}{x^3-x}\,dx = \int \frac{1}{x} + \frac{2}{x-1} - \frac{2}{x+1}\,dx &= \ln|x| + 2\ln|x-1| - 2\ln|x+1| + c\\ &= \ln|x| + 2\ln\left|\frac{x-1}{x+1}\right| + c.\end{aligned}$$

EXAMPLE 7.3.3 (Partial Fractions: Three Distinct Linear Factors). Determine

$$\int \frac{4x+28}{(x+1)(x^2-4x+3)}\,dx.$$

SOLUTION. Check that this is not a substitution integral. However, the degree of the numerator is less than the degree of the denominator $(1 < 3)$ and the denominator factors into distinct three linear factors: $(x+1)(x^2-4x+3) = (x+1)(x-1)(x-3)$. We form the partial fractions with a different constant for each linear factor:

$$\begin{aligned}\frac{4x+28}{(x+1)(x^2-4x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-3} &= \frac{A(x^2-4x+3)+B(x^2-2x-3)+C(x^2-1)}{(x+1)(x^2-4x+3)}\\ &= \frac{(A+B+C)x^2+(-4A-2B)x+3A-3B-C}{(x+1)(x^2-4x+3)}.\end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{lrcrll} x^2\text{'s:} & 0 & = & A+B+C & & \Rightarrow\ C=5 \\ x\text{'s:} & 4 & = & -4A-2B & \Rightarrow\ A=3 & \\ \text{const:} & 28 & = & 3A-3B-C & & \\ \hline \text{Add all:} & 32 & = & -4B & \Rightarrow B=-8 & \end{array}$$

The integration becomes:

$$\begin{aligned}\int \frac{4x+28}{(x+1)(x^2-4x+3)}\,dx &= \int \frac{3}{x+1} - \frac{8}{x-1} + \frac{5}{x-3}\,dx\\ &= 3\ln|x-1| - 8\ln|x+1| + 5\ln|x-3| + c\end{aligned}$$

YOU TRY IT 7.4. What would the numerator in the original integral have to be to make the problem a substitution problem?

7.4 A Second Complication: Repeated Factors

Another complication that can arise is that the denominator of the rational function factors into linear factors, but the factors are not distinct. That is, there are repeated linear factors. The solution in this situation is to *include a term for every power of every linear factor that divides the denominator*.

EXAMPLE 7.4.1 (Partial Fractions: Repeated Linear Factors). Determine

$$\int \frac{3x^2 - 7x + 2}{x^3 - 2x^2 + x}\, dx.$$

SOLUTION. This is not a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into repeated linear factors:

$$x(x^2 - 2x + 1) = x(x-1)(x-1) = x(x-1)^2.$$

We form the partial fractions with a different constant for each power of each factor that divides the denominator. Note: There are terms for both $x - 1$ and $(x-1)^2$. Warning! Look very carefully at the numerator in the step where the common denominator is formed.

$$\begin{aligned} \frac{3x^2 - 7x + 2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} &= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} \\ &= \frac{Ax^2 - 2Ax + A + Bx^2 - Bx + Cx}{x(x-1)^2}. \end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{llcll} x^2\text{'s:} & 3 & = & A + B & \qquad \Rightarrow 1 = B \\ x\text{'s:} & -7 & = & -2A - B + C & \qquad\qquad \Rightarrow C = -2 \\ \text{const:} & 2 & = & A \quad \Rightarrow A = 2 & \end{array}$$

Notice that we used the value $A = 2$ to simplify the first two equations. The integration becomes:

$$\begin{aligned} \int \frac{3x^2 - 7x + 2}{x^3 - 2x^2 + x}\, dx &= \int \frac{2}{x} + \frac{1}{x-1} - \frac{2}{(x-1)^2}\, dx \\ &= 2\ln|x| + \ln|x-1| + 2(x-1)^{-1} + c. \end{aligned}$$

Be careful! The final integral requires the power rule and a mini-substitution.

YOU TRY IT 7.5. What would the numerator in the original integral have to be to make the problem a substitution problem?

EXAMPLE 7.4.2 (Partial Fractions: Repeated Linear Factors). Determine

$$\int \frac{3x^2 - 2x - 3}{x^3 - x^2}\, dx.$$

SOLUTION. This is not quite a substitution integral. However, the degree of the numerator is less than the degree of the denominator ($2 < 3$) and the denominator factors into repeated linear factors: $x^2(x-1)$. We form the partial fractions with a different constant for each power of each factor that divides the denominator—there are terms for both x and x^2. Look very carefully at the numerator in the step where the common denominator is formed.

$$\begin{aligned} \frac{3x^2 - 2x - 3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} &= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)} \\ &= \frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)}. \end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{llcll} x^2\text{'s:} & 3 & = & A+C & \Rightarrow C=-2 \\ x\text{'s:} & -2 & = & -A+B & \Rightarrow A=5 \\ \text{const:} & -3 & = & -B & \Rightarrow B=3 \end{array}$$

Notice that we used the value $B = 3$ to simplify the first two equations. The integration becomes:

$$\int \frac{3x^2-2x-3}{x^3-x^2}\,dx = \int \frac{5}{x} + \frac{3}{x^2} - \frac{2}{x-1}\,dx = 5\ln|x| - 3x^{-1} - 2\ln|x-1| + c.$$

YOU TRY IT 7.6. What would the numerator in the original integral have to be to make the problem a substitution problem?

YOU TRY IT 7.7. Determine

$$\int \frac{7-x}{(x+1)(x-1)^2}\,dx.$$

Answer to YOU TRY IT 7.7 : $2\ln|x+1| - 2\ln|x-1| - \frac{3}{x-1}$.

EXAMPLE 7.4.3 (Partial Fractions: Repeated Linear Factors). Here's a different one: Determine

$$\int \frac{x^2}{(x+1)^3}\,dx.$$

SOLUTION. The degree of the numerator is less than the degree of the denominator $(2 < 3)$ and the denominator factors into repeated linear factors: $(x+1)^3 = (x+1)(x+1)(x+1)$. We form the partial fractions with a different constant for each power of each factor that divides the denominator: $(x+1)$, $(x+1)^2$, and $(x+1)^3$. Look very carefully at the numerator in the step where the common denominator is formed.

$$\begin{aligned} \frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} &= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} \\ &= \frac{Ax^2 + 2Ax + A + Bx + B + C}{(x+1)^3}. \end{aligned}$$

Compare like terms in the numerators of the first and last functions.

$$\begin{array}{llcll} x^2\text{'s:} & 1 & = & A & \Rightarrow A=1 \\ x\text{'s:} & 0 & = & 2A+B & \Rightarrow B=-2 \\ \text{const:} & 0 & = & A+B+C & \Rightarrow C=1 \end{array}$$

Notice that we used the value $A = 1$ to simplify the last two equations. The integration becomes:

$$\begin{aligned} \int \frac{x^2}{(x+1)^3}\,dx &= \int \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\,dx \\ &= \ln|x+1| + 2(x+1)^{-1} - \tfrac{1}{2}(x+1)^{-2} + c. \end{aligned}$$

Alternative Solution. We could have used a u-substitution to solve this problem. Let $u = x+1$. Then $du = dx$. Since $u = x+1$, then $x = u-1$ so $x^2 = (u-1)^2 = u^2 - 2u + 1$. Therefore

$$\begin{aligned} \int \frac{x^2}{(x+1)^3}\,dx = \int \frac{u^2-2u+1}{u^3}\,du &= \int \frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\,du \\ &= \ln|u| - 2u^{-1} - \tfrac{1}{2}u^{-2} + c \\ &= \ln|x+1| + 2(x+1)^{-1} - \tfrac{1}{2}(x+1)^{-2} + c, \end{aligned}$$

just as above. Which method would you have used? Why?

7.5 Problems

1. Try integrating these rational functions (answers below). These are a bit harder than those in the text. Some have three factors. Others have repeated factors.

(*a*) $\displaystyle\int \frac{4t^2-3t-4}{t^3-t^2-2t}\,dt$ (*b*) $\displaystyle\int \frac{t+7}{(t+1)(t^2-4t+3)}\,dt$ (*c*) $\displaystyle\int \frac{x+6}{x^2-x-6}\,dx$

(*d*) $\displaystyle\int \frac{x}{(x-1)(x+1)(x+2)}\,dx$ (*e*) $\displaystyle\int \frac{2x^2}{(x-1)^2(x+1)}\,dx$ (*f*) $\displaystyle\int \frac{-4x+4}{(x-2)^2x}\,dx$

(*g*) $\displaystyle\int \frac{2x}{(x-2)(x^2-1)}\,dx$ (*h*) $\displaystyle\int \frac{2x-1}{(x+2)^2}\,dx$

2. Try these similar looking problems.

(*a*) $\displaystyle\int \frac{10}{25+x^2}\,dx$ (*b*) $\displaystyle\int \frac{10x}{25+x^2}\,dx$ (*c*) $\displaystyle\int \frac{10}{25-x^2}\,dx$

3. Have you finished all the ones above? Do these similar looking integrals.

(*a*) $\displaystyle\int \frac{4}{\sqrt{4+x^2}}\,dx$ (*b*) $\displaystyle\int \frac{4}{\sqrt{4-x^2}}\,dx$

(*c*) $\displaystyle\int \frac{4x}{(4+x^2)^{3/2}}\,dx$ (*d*) $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$

Answers to Practice Problems

1. (*a*) $\int \frac{2}{t} + \frac{1}{t-2} + \frac{1}{t+1}\,dt = 2\ln|t| + \ln|t-2| + \ln|t+1| + c$

(*b*) $\int \frac{3/4}{t+1} + \frac{5/4}{t-3} - \frac{2}{t-1}\,dt = \frac{3}{4}\ln|t+1| + \frac{5}{4}\ln|t-3| - 2\ln|t-1| + c$

(*c*) $\int \frac{9/5}{x-3} - \frac{4/5}{x+2}\,dt = x + \frac{9}{5}\ln|x-3| - \frac{4}{5}\ln|x+2| + c$

(*d*) $\int \frac{1/6}{x-1} + \frac{1/2}{x+1} - \frac{2/3}{x+2}\,dx = \frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{2}{3}\ln|x+2| + c$

(*e*) $\int \frac{3/2}{x-1} + \frac{1}{(x-1)^2} + \frac{1/2}{x+1}\,dx = \frac{3}{2}\ln|x-1| - (x-1)^{-1} + \frac{1}{2}\ln|x+1| + c$

(*f*) $\int -\frac{1}{x-2} - \frac{2}{(x-2)^2} + \frac{1}{x}\,dx = -\ln|x-2| + 2(x-2)^{-1} + \ln|x| + c$

(*g*) $\int \frac{2}{x+2} - \frac{5}{(x+2)^2}\,dx = 2\ln|x+2| + \frac{5}{x+2} + c.$

2. All "$+c$".

(*a*) $2\arctan(x/5)$ (*b*) $5\ln|25+x^2|$ (*c*) $\ln\left|\dfrac{x+5}{x-5}\right|$