Series: Infinite Sums

Series are a way to make sense of certain types of infinitely long sums. We will need to be able to do this if we are to attain our goal of approximating transcendental functions by using 'infinite degree' polynomials. But before we try to add together an infinite number of polynomials, we first explore what it means to add an infinite number of numbers.

Here's the issue: We know how to add two numbers: $a_1 + a_2$. Using associativity (and parentheses) we can add three numbers

$$a_1 + (a_2 + a_3)$$

four numbers

$$a_1 + (a_2 + (a_3 + a_4))$$

or even n numbers

$$a_1 + (a_2 + (a_3 + (a_4 + (\cdots + (a_{n-1} + a_n)\ldots)))).$$

But where would we start (or end) when trying to add an infinite number of terms? And does the sum add up to a finite number or not? Since all we know how to do is add a finite number of terms, we will have to use finite addition and limits to make sense of the process.

Introduction to Series

OK, enough of this finite stuff. What we want to do is add up the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$. More precisely, given a sequence $\{a_n\}_{n=1}^{\infty}$, we can form the infinite sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

which is called an **infinite series** or more simply just a **series**.

Can we do this? Here are several examples.

(*a*) $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \cdots = \infty$. The sum is clearly not finite; the series diverges.

(*b*) $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. Do these terms add up to a finite sum?

(*c*) $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Do these terms add up to a finite sum?

(*d*) $\sum_{n=0}^{\infty} \frac{1}{(-2)^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$. Do these terms add up to a finite sum?

DEFINITION 13.1. To find the sum of an infinite series $\sum_{k=1}^{\infty} a_k$ we form the **sequence of partial sums** that are often denoted by S_n.

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \qquad (S_n \text{ is called the } n\textbf{th partial sum of the series}) \end{aligned}$$

If the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ has a limit a limit L (converges), we say that the series **converges to** L and we write:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L$$

or just

$$\sum_{k=1}^{\infty} a_k = L.$$

Otherwise the series **diverges**.

EXAMPLE 13.1. Here's a simple example. Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + + \frac{1}{8} + \frac{1}{16} + \cdots,$$

if it exists.

SOLUTION. We first determine each partial sum and then rewrite it in a more convenient form.

$$\begin{aligned} S_1 &= \frac{1}{2} = 1 - \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4} \\ S_2 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8} \\ &\vdots \\ S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} = 1 - \frac{1}{2^n} \end{aligned}$$

So the sequence of partial sums is $\{S_n\}_{n=1}^{\infty} = \left\{1 - \frac{1}{2^n}\right\}_{n=1}^{\infty}$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1 - 0 = 1,$$

where we have used Theorem 13.2 to evaluate the limit. In other words,

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Pretty cool!

EXAMPLE 13.2. Here's a another fun example. Find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k}$ if it exists.

SOLUTION. Using partial fractions (check this)

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

Notice that most of the terms cancel out. The sum collapses and we see that

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Such a sum is called a **telescoping sum**. We are left with only the first and last terms in the partial sum. This time

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = 1.$$

In other words,

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1.$$

YOU TRY IT 13.1. Try this telescoping sum. Find the sum of the series $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$ if it exists. This time there will be a few more terms that do not cancel. See if you can figure it out.

EXAMPLE 13.3 (Partial Fractions). Here's another example that uses **partial fractions**. Find the sum of the series $\sum_{k=0}^{\infty} \frac{4}{k^2+3k+2}$ if it exists.

SOLUTION. Since the degree of the numerator is smaller than the degree of the denominator a nd since the denominator factors into linear factors, we can write

$$\frac{4}{k^2+3k+2} = \frac{4}{(k+1)(k+2)} = \frac{A}{k+1} + \frac{B}{k+2} = \frac{Ak + 2A + Bk + B}{(k+1)(k+2)}.$$

Solving we get:

$$k\text{'s}: \quad 0 = A + B. \tag{13.1}$$

and

$$\text{constants}: \quad 4 = 2A + B. \tag{13.2}$$

Subtracting (13.1) from (13.2) gives

$$4 = A. \tag{13.3}$$

Putting $A = 4$ in (13.1) makes $B = -4$. So we see that

$$\frac{4}{k^2+3k+2} = \frac{4}{k+1} - \frac{4}{k+2}.$$

(Check that this is correct!) This means that

$$\sum_{k=0}^{\infty} \frac{4}{k^2+3k+2} = \sum_{k=0}^{\infty} \left(\frac{4}{k+1} - \frac{4}{k+2}\right)$$

which is another telescoping series. This time

$$S_n = \left(\frac{4}{1} - \frac{4}{2}\right) + \left(\frac{4}{2} - \frac{4}{3}\right) + \left(\frac{4}{3} - \frac{4}{4}\right) + \cdots + \left(\frac{4}{n+1} - \frac{4}{n+2}\right) = 4 - \frac{4}{n+2}.$$

So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 4 - \frac{4}{n+2} = 4 - 0 = 4.$$

In other words,

$$\sum_{k=0}^{\infty} \frac{4}{k^2+3k+2} = 4.$$

Wow!

EXAMPLE 13.4 (Telescoping). Here's a more complicated example that uses partial fractions. Find the sum of the series $\sum_{k=1}^{\infty} \ln \frac{(k+1)}{k}$ if it exists.

SOLUTION. We can use a log property to rewrite the partial sum as

$$\begin{aligned} S_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) &= \sum_{k=1}^{n} \ln(k+1) - \ln k \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + [\ln(n+1) - \ln n] \\ &= \ln n - \ln 1. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln n - \ln 1 = \infty \text{ (diverges)}$$

and the series

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$$

diverges.

EXAMPLE 13.5 (Partial Fractions). Here's a more complicated example that uses partial fractions. Find the sum of the series $\sum_{k=0}^{\infty} \frac{8}{k^2+4k+3}$ if it exists.

SOLUTION. Since the degree of the numerator is smaller than the degree of the denominator a nd since the denominator factors into linear factors, we can write

$$\frac{8}{k^2+4k+3} = \frac{4}{(k+1)(k+3)} = \frac{A}{k+1} + \frac{B}{k+3} = \frac{Ak+3A+Bk+B}{(k+1)(k+3)}.$$

Solving we get:

$$k\text{'s}: \quad 0 = A + B. \tag{13.4}$$

and

$$\text{constants}: \quad 8 = 3A + B. \tag{13.5}$$

Subtracting (13.4) from (13.5) gives

$$8 = 2A. \tag{13.6}$$

Putting $A = 4$ in (13.4) makes $B = -4$. So we see that

$$\frac{8}{k^2+4k+3} = \frac{4}{k+1} - \frac{4}{k+3}.$$

This means that

$$\sum_{k=0}^{\infty} \frac{8}{k^2+4k+3} = \sum_{k=0}^{\infty} \left(\frac{4}{k+1} - \frac{4}{k+3}\right)$$

which is another telescoping series.

$$\begin{aligned} S_n &= \left(\tfrac{4}{1} - \tfrac{4}{3}\right) + \left(\tfrac{4}{2} - \tfrac{4}{4}\right) + \left(\tfrac{4}{3} - \tfrac{4}{5}\right) + \left(\tfrac{4}{4} - \tfrac{4}{5}\right) + \cdots \\ &\qquad \cdots + \left(\tfrac{4}{n-1} - \tfrac{4}{n+1}\right) + \left(\tfrac{4}{n} - \tfrac{4}{n+2}\right) + \left(\tfrac{4}{n+1} - \tfrac{4}{n+3}\right) \\ S_n &= 4 + 2 - \tfrac{4}{n+2} - \tfrac{4}{n+3}. \end{aligned}$$

So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 6 - \frac{4}{n+2} - \frac{4}{n+3} = 6.$$

In other words,

$$\sum_{k=0}^{\infty} \frac{8}{k^2+4k+3} = 6.$$

YOU TRY IT 13.2 (Partial fractions). Here are two others that are similar to the last example in that they use partial fractions. See if you can solve them. Find the sums of these series if they exist.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2+7k+12}$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2+4k+3}$

Answers: (a) $\frac{1}{4}$; (b) $\frac{3}{2}$.

Geometric Series

Geometric series are among the simpler with which to work. We will see that we can determine which ones converge and what their limits are fairly easily.

DEFINITION 13.2. A **geometric series** is a series that has the form $\sum_{n=0}^{\infty} ar^n$, where a is a real constant and r is a real number.

YOU TRY IT 13.3. Here are a few examples. Identify a and r in each.

$(a)\ \sum_{n=0}^{\infty} 6 \cdot 4^n \qquad (b)\ \sum_{n=0}^{\infty} \frac{1}{2^n} \qquad (c)\ \sum_{n=0}^{\infty} 2 \cdot 3^{-n} \qquad (d)\ \sum_{n=2} 5 \cdot \left(-\frac{2}{3}\right)^n \qquad (e)\ \sum_{n=1} 2 \cdot \left(\frac{1}{3}\right)^{2n}$

Answers: (a) 6 and 4; (b) 1 and 1/2; (c) 2 and 1/3; (d) $a = 20/9$ and $r = -2/3$; (e) $a = 2/9$ and $r = 1/9$.

Determining the sum of a geometric series $\sum_{n=0}^{\infty} ar^n$ is relatively simple. We begin by comparing the nth partial sum S_n with rS_n. We find:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^n \tag{13.7}$$

$$rS_n = \quad ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1} \tag{13.8}$$

So subtracting (13.8) from (13.7) we obtain

$$S_n - rS_n = a - ar^{n-1}$$

or

$$(1-r)S_n = a(1 - r^{n+1}).$$

So

$$S_n = \frac{a(1-r^{n+1})}{1-r}. \tag{13.9}$$

We know from the Key Limit Theorem 13.2 that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{diverges} & \text{otherwise.} \end{cases} \tag{13.10}$$

Thus, putting (13.9) and (13.10) together we find

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{otherwise.} \end{cases}$$

So we have proved

THEOREM 13.1 (Geometric Series Test)**.** If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ diverges.

EXAMPLE 13.6. Here are some examples that get progressively more complex.

(*a*) Find the sum of the series $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$ if it exists.

SOLUTION. In this example $a = 1$ and $r = \frac{2}{5}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{1}{1-\frac{2}{5}} = \frac{5}{3}$.

(*b*) Find the sum of the series $\sum_{n=0}^{\infty} 4\left(\frac{6}{7}\right)^n$ if it exists.

SOLUTION. In this example $a = 4$ and $r = \frac{6}{7}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{4}{1-\frac{6}{7}} = \frac{28}{1} = 28$.

(*c*) Find the sum of the series $\sum_{n=0}^{\infty} 2\left(\frac{3}{2}\right)^n$ if it exists.

SOLUTION. In this example $a = 2$ and $r = \frac{3}{2}$. Since $|r| > 1$, by Theorem 13.1 the series diverges.

(*d*) Find the sum of the series $\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^{n+2}$ if it exists.

SOLUTION. Before we can apply the Geometric Series Test, we have to adjust the power. Notice that we can rewrite the series using the nth power using

$$\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^{n+2} = \sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^{2}\left(-\frac{1}{2}\right)^{n} = \sum_{n=0}^{\infty} \frac{5}{4}\left(-\frac{1}{2}\right)^{n}.$$

Now $a = \frac{5}{4}$ and $r = -\frac{1}{2}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{\frac{5}{4}}{1-(-\frac{1}{2})} = \frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{6}$.

SOLUTION. Alternative Method. Another way that we can approach this problem is to write out the first few terms of the series and identify a and r.

$$\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^{n+2} = \underbrace{\frac{5}{4}}_{a} - \underbrace{\frac{5}{8}}_{ar} + \underbrace{\frac{5}{16}}_{ar^2} - \underbrace{\frac{5}{32}}_{ar^3} + \cdots.$$

Now $a = \frac{5}{4}$ and the ratio of a term to the previous one is $r = -\frac{1}{2}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{\frac{5}{4}}{1-(-\frac{1}{2})} = \frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{6}$. I find this easier!

(*e*) Find the sum of the series $\sum_{n=2} 2\left(\frac{1}{3}\right)^n$ if it exists.

SOLUTION. We use the **Alternative Method.** Write out the first few terms of the series and identify a and r.

$$\sum_{n=2} 2\left(\frac{1}{3}\right)^n = \underbrace{\frac{2}{9}}_{a} + \underbrace{\frac{2}{27}}_{ar} + \underbrace{\frac{2}{81}}_{ar^2} + \underbrace{\frac{2}{243}}_{ar^3} + \cdots.$$

Now $a = \frac{2}{9}$ and the ratio of a term to the previous one is $r = \frac{1}{3}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{\frac{2}{9}}{1-(\frac{1}{3})} = \frac{\frac{2}{9}}{\frac{2}{3}} = \frac{1}{3}$.

Some series start at $n = 1$ or $n = 2$ or some other index $n \neq 0$. For instance, consider the general series $\sum_{n=2}^{\infty} a_n$. We can write it as a series starting at $n = 0$ if we are careful to also subtract off the first two terms:

$$\sum_{n=2}^{\infty} a_n = \left(\sum_{n=0}^{\infty} a_n\right) - (a_0 + a_1).$$

We can apply this to geometric series.

EXAMPLE 13.7. Here are a two more examples.

(*a*) Find the sum of the series $\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$ if it exists.

SOLUTION. First rewrite the series adding back and then subtracting the first few 'missing' terms.

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \left(\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n\right) - \left(1 + \frac{2}{3} + \frac{4}{9}\right).$$

By Theorem 13.1 the series converges to $\frac{1}{1-\frac{2}{3}} - \left(1 + \frac{2}{3} + \frac{4}{9}\right) = 3 - \frac{19}{9} = \frac{8}{9}$.

Alternative Method. Write out the first few terms of the series and identify a and r.

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \underbrace{\frac{8}{27}}_{a} + \underbrace{\frac{16}{81}}_{ar} + \underbrace{\frac{32}{243}}_{ar^2} + \underbrace{\frac{64}{729}}_{ar^3} + \cdots.$$

Now $a = \frac{8}{27}$ and the ratio of a term to the previous one is $r = \frac{2}{3}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{\frac{8}{27}}{1-(\frac{2}{3})} = \frac{8}{9}$.

(*b*) Find the sum of the series $\sum_{n=2}^{\infty} 4\left(-\frac{1}{3}\right)^n$ if it exists.

SOLUTION. First rewrite the series; be careful of the signs.

$$\sum_{n=2}^{\infty} 4\left(-\frac{1}{3}\right)^n = \left(\sum_{n=0}^{\infty} 4\left(-\frac{1}{3}\right)^n\right) - \left(4 - \frac{4}{3}\right).$$

By Theorem 13.1 the series converges to $\frac{4}{1-(-\frac{1}{3})} - \left(4 - \frac{4}{3}\right) = 3 - \frac{8}{3} = \frac{1}{3}$.

Alternative Method. Write out the first few terms of the series and identify a and r.

$$\sum_{n=2}^{\infty} 4\left(-\frac{1}{3}\right)^n = \underbrace{\frac{4}{9}}_{a} - \underbrace{\frac{4}{27}}_{ar} + \underbrace{\frac{4}{81}}_{ar^2} - \underbrace{\frac{4}{243}}_{ar^3} + \cdots.$$

Now $a = \frac{4}{9}$ and the ratio of a term to the previous one is $r = \frac{1}{3}$ and $|r| < 1$. So by Theorem 13.1 the series converges to $\frac{\frac{4}{9}}{1-(-\frac{1}{3})} = \frac{1}{3}$.

(Optional) Application: Repeating Decimals

You may (or may not) remember from your high school math days that every repeating decimal can be expressed as a **rational number**, that is, as a fraction using integers. There are some familiar ones such as

$$0.3333\cdots = 0.\overline{3} = \frac{1}{3}.$$

Similarly we have

$$0.6666\cdots = 0.\overline{6} = \frac{2}{3}$$

Note: We use a horizontal bar to indicate which part of the decimal repeats.

and

$$0.1111\cdots = 0.\overline{1} = \frac{1}{9}.$$

But what about something like $0.\overline{12} = 0.121212\ldots$? We can write any such expression as a geometric series. In this case

$$\begin{aligned}
0.\overline{12} = 0.121212\ldots &= 0.12 + 0.0012 + 0.000012 + \cdots \\
&= \frac{12}{(10)^2} + \frac{12}{(10)^4} + \frac{12}{(10)^6} + \frac{12}{(10)^8} + \cdots \\
&= \frac{12}{(10)^2}\left(1 + \frac{1}{(10)^2} + \frac{1}{(10)^4} + \frac{1}{(10)^6} + \cdots\right) \\
&= \frac{12}{(10)^2}\left(1 + \frac{1}{(10)^2} + \left(\frac{1}{(10)^2}\right)^2 + \left(\frac{1}{(10)^2}\right)^3 + \cdots\right) \\
&= \frac{12}{100}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots\right) \\
&= \sum_{n=0}^{\infty} \frac{12}{100}\left(\frac{1}{100}\right)^n \\
&= \frac{\frac{12}{100}}{1 - \frac{1}{100}} \\
&= \frac{\frac{12}{100}}{\frac{99}{100}} \\
&= \frac{12}{99}.
\end{aligned}$$

So $0.\overline{12} = 0.121212\ldots$ is rational since it can be written as the fraction $\frac{12}{99}$.

EXAMPLE 13.8. Here are a few more repeating decimals to try.

Answers: (b) $\frac{123}{999}$; (c) $\frac{abcd}{9999}$; (d) $\frac{1222}{9900}$.

(*a*) Express $0.\overline{9} = 0.9999\ldots$ as a rational number. What is interesting about the answer?

(*b*) Express $0.\overline{123}$ as a rational number.

(*c*) Express $0.\overline{abcd}$ as a rational number, where a, b, c, and d are nonnegative integers.

(*d*) Express $0.12\overline{34}$ as a rational number.

Basic Properties of Series

The Algebra of Series

There are several simple properties that apply to series that converge. We have already been using the first of these results.

THEOREM 14.1. Suppose that we have two convergent series: $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$. Then

(*a*) If c is a constant, then $\sum_{n=0}^{\infty} ca_n = cA$.

(*b*) $\sum_{n=0}^{\infty} a_n \pm b_n = A \pm B$.

EXAMPLE 14.1. Determine the sum of the series $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n + 3\left(-\frac{1}{2}\right)^n$ if it exists.

SOLUTION. Using the Geometric Series Test (Theorem 13.1) and Theorem 14.1 we have

$$\begin{aligned}\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n + 3\left(-\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} 3\left(-\frac{1}{2}\right)^n \\ &= \frac{3}{1-\frac{1}{2}} + \frac{3}{1-(-\frac{1}{2})} \\ &= 6 + 2 = 8.\end{aligned}$$

The Divergence Test

The next theorem shows that for a series to converge, the terms of the series must get small as n gets large.

THEOREM 14.2 (The nth term Test). If $\sum_{n=0}^{\infty} a_n$ converges to A, then $\lim_{n\to\infty} a_n = 0$.

Proof. We make use of a comment we made earlier. Let S_n be the nth partial sum of the series. We know that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = A$ because both sequences are really just the same set of numbers. But

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

and

$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}.$$

So

$$S_n - S_{n-1} = a_n.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = A - A = 0,$$

which is what we wanted to prove. □

The theorem is seldom used in the form as stated above. Rather, if the nth term of a series does *not* converge to 0, then series cannot convergence. This is a more useful way of stating of Theorem 14.2 and in this form it is a test for *divergence*.

THEOREM 14.3 (The nth term test for divergence). If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Warning! The nth term test for divergence never allows us to conclude that a series converges, only that it does not converge. If $\lim_{n\to\infty} a_n = 0$, we can't conclude anything. The series may or may not converge. The test simply fails to provide us with any useful information in such a case.

EXAMPLE 14.2. Determine whether the series $\sum_{n=0}^{\infty} \frac{n^2+1}{3n^2+n+1}$ converges.

SOLUTION. Notice that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{3n^2+n+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{3+\frac{1}{n}+\frac{1}{n}} = \frac{1}{3} \neq 0.$$

By the nth term test for divergence (Theorem 14.3), the series $\sum_{n=0}^{\infty} \frac{n^2+1}{3n^2+n+1}$ diverges.

EXAMPLE 14.3. Determine whether the series $\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$ converges.

SOLUTION. This time using using one of our key limits (see Theorem 13.2)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{k}{n}\right)^n = e^k \neq 0.$$

By the nth term test for divergence (Theorem 14.3), the series $\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$ diverges.

EXAMPLE 14.4. Determine whether the series $\sum_{n=1}^{\infty} \sqrt[n]{n}$ converges.

SOLUTION. Using another of our key limits (see Theorem 13.2)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = 1 \neq 0.$$

By the nth term test for divergence (Theorem 14.3), the series $\sum_{n=1}^{\infty} \sqrt[n]{n}$ diverges.

EXAMPLE 14.5. Determine whether the series $\sum_{k=1}^{\infty} \frac{2^n}{n^2} = \frac{2}{1} + \frac{4}{4} + \frac{8}{9} + \cdots$ converges.

SOLUTION. Notice that both numerator and denominator both tend to infinity. So converting to x and using l'Hôpital's rule (twice!)

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)(\ln 2)2^x}{2} = \infty.$$

By the nth term test for divergence (Theorem 14.3), the series diverges.

We will need to use the derivative formula $\frac{d}{dx}(a^x) = a^x \ln a$, which is valid when $a > 0$.

EXAMPLE 14.6. Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{n}$ and $\sum_{k=1}^{\infty} \frac{1}{n^2}$ converge.

SOLUTION. Notice that both
$$\lim_{n\to\infty} \frac{1}{n} = 0.$$
and
$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$
In this situation the nth term test for divergence (Theorem 14.3) **fails** to provide us with any information. The series may or may not converge. In fact, we will soon see that one of these series converges while the other diverges.

EXAMPLE 14.7. Determine whether the series $\sum_{k=1}^{\infty} \cos \frac{1}{n}$ converges.

SOLUTION. Notice that

$$\lim_{n\to\infty} \cos \tfrac{1}{n} = \cos 0 = 1 \neq 0.$$

By the nth term test for divergence (Theorem 14.3) the series diverges.

YOU TRY IT 14.4. Here are six series. Which of them can you say diverge by the nth term test for **divergence**? For which series is this test not helpful. Explain.

(*a*) $\sum_{n=1}^{\infty} \frac{3n+1}{2n+5}$ (*b*) $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ (*c*) $\sum_{n=1}^{\infty} (1.1)^n$

(*d*) $\sum_{n=1}^{\infty} \left(1+\frac{4}{n}\right)^n$ (*e*) $\sum_{n=1}^{\infty} \sqrt[n]{n}$ (*f*) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$