The Comparison Tests

The idea behind the comparison tests is pretty simple. Suppose we have a series such as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which we know converges by the p-series test. Now compare this to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n+2}$. The terms in this new series are *smaller* than the corresponding terms the first series since

$$0 < \frac{1}{n^2+n+2} < \frac{1}{n^2}.$$

So the sum of $\sum_{n=1}^{\infty} \frac{1}{n^2+n+2}$ should be smaller than the sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the latter series converges, so does the former.

There are some technical details that need to be checked—for instance, the terms of the series need to be non-negative. But this idea can be made into a proof which we will omit here. The result is

THEOREM 14.6 (The Direct Comparison Test). Assume that $0 \le a_n \le b_n$ for all n (or at least all $n \ge k$).

1. If $\sum_{n=1}^{\infty} b_n$ converges so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ diverges so does $\sum_{n=1}^{\infty} b_n$.

NOTE: As stated, this test requires $0 \le a_n \le b_n$ for all n. But this condition may be relaxed so that $0 \le a_n \le b_n$ for all $n > k$.

The way to think about this theorem is if the bigger series converges, so does the smaller one. If the smaller one diverges, then so does the bigger one.

Pre-law and pre-med students (in addition to math students) should delight in using the the direct comparison test because one needs to see a pattern and then construct a little argument. I will be looking for these 'arguments' when I grade your work.

Examples

To use the comparison test effectively, you need to know lots of series that diverge or converge to which you can compare an unknown series. Such series are often 'provided' by such tests as the p-series test, the geometric series test, or even the integral series test. Let's see how this works.

EXAMPLE 14.15. Does $\sum_{n=1}^{\infty} \frac{1}{2^n+5}$ converge?

SOLUTION. SCRAP WORK: Notice this not a series to which the integral test easily applies, nor is it a p-series or a geometric series. However, it looks a lot like the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ which converges. When we compare a series to a converging series, we want the unknown series to be *smaller* than the known series so that we can use the the first part of the the direct comparison test to show that the new series also converges. In this case notice that $0 \le \frac{1}{2^n+5} < \frac{1}{2^n}$ for all n. OK, let's give a careful argument.

ARGUMENT: Since $0 \le \frac{1}{2^n+5} \le \frac{1}{2^n}$ for all n, and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by the geometric series test (Theorem 13.1) because $|r| = \frac{1}{2} < 1$, then $\sum_{n=1}^{\infty} \frac{1}{2^n+5}$ converges by the direct comparison test (Theorem 14.6).

Notice the argument is not long but it has two important aspects. First we identified a related series that we knew about for comparison. Second we verified the appropriate hypothesis relating the terms of the unknown series to the one we knew about.

EXAMPLE 14.16. Does $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

SOLUTION. SCRAP WORK: We could use the integral test. However, this looks a lot like the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. When we compare to a diverging series, we are expecting the unknown series to diverge so we want the terms of the unknown series to be *larger* than the terms of the known series. But is it true that $\frac{\ln n}{n} > \frac{1}{n}$? OK, let's give a careful argument.
ARGUMENT: Notice that

$$\frac{\ln n}{n} \geq \frac{1}{n} \iff \ln n \geq 1 \iff n \geq e.$$

In particular, if $n \geq 3$, then $\frac{\ln n}{n} \geq \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series test with $p = 1$), then $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the direct comparison test (Theorem 14.6).

Notice how we justified the steps in the argument, even justifying why we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

EXAMPLE 14.17. Does $\sum_{n=1}^{\infty} \frac{2^n}{5^n + 6}$ converge?

SOLUTION. SCRAP WORK: This looks a lot like the geometric series $\sum_{n=1}^{\infty} \frac{2^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ which converges.
ARGUMENT: Since $0 \leq \frac{2^n}{5^n+6} \leq \left(\frac{2}{5}\right)^n$ for all n, and since $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ converges by the geometric series test (Theorem 13.1) because $|r| = \frac{2}{5} < 1$, then $\sum_{n=1}^{\infty} \frac{2^n}{5^n + 6}$ converges by the direct comparison test (Theorem 14.6).

EXAMPLE 14.18. Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge?

SOLUTION. SCRAP WORK: Notice that the terms of this series get small very quickly. So we should suspect that it converges. This not a series to which the integral test easily applies, nor is it a p-series. It takes a bit of algebra to see what to compare it to.
ARGUMENT: Notice $n! = 1 \cdot 2 \cdots (n-1) \cdot n \geq (n-1) \cdot n = n^2 - n$. So $0 < \frac{1}{n!} \leq \frac{1}{n^2-n}$. Now we could apply the integral test to the series $\sum \frac{1}{n^2-n}$ to see that it converges, and then use the direct comparison test to see that $\sum \frac{1}{n!}$ converges. But we can avoid the integral test by using a bit more algebra. Notice that

$$n^2 - n \geq \frac{n^2}{2} \iff \frac{n^2}{2} \geq n \iff \frac{n}{2} \geq 1 \iff n \geq 2.$$

So this means $n! \geq \frac{n^2}{2}$ if $n \geq 2$. So $\frac{1}{n!} < \frac{2}{n^2}$, when $n \geq 2$. However $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges (p-series, $p = 2 > 1$) so by direct comparison (Theorem 14.6) the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

YOU TRY IT 14.6. Each of the following statements is an attempt to show that a given series is convergent or divergent using the Comparison Test. Classify each statement, 'correct' if the argument is valid, or 'incorrect' if any part of the argument is flawed. (Note: Even if the conclusion is true but the argument that led to it was wrong, classify it as incorrect.)

(*a*) For all $n > 3$, $0 \le \frac{1}{n} \le \frac{1}{n\ln(n)}$, and the series $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges so by the Comparison Test, the series $\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$ diverges.

(*b*) For all $n > 2$, $0 \le \frac{1}{n} < \frac{\sqrt{n+1}}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$ diverges.

(*c*) For all $n > 2$, $0 \le \frac{n}{3-n^3} < \frac{1}{n^2}$, and the series $\sum_{n=2} \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum_{n=2} \frac{n}{3-n^3}$ converges.

(*d*) For all $n \ge 1$, $0 \le \frac{\cos^2(n)}{n^3} < \frac{1}{n^3}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so by the Comparison Test, the series $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3}$ converges.

(*e*) For all $n \ge 1$, $0 \le \frac{1}{n^2} < \frac{2n+1}{n^3}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^3}$ converges.

Answers to YOU TRY IT 14.6 : (a) Incorrect: $\frac{1}{n} \not\le \frac{1}{n\ln(n)}$. (b) Correct. (c) Incorrect. $0 \not\le \frac{n}{3-n^3}$ (d) Correct. (e) Incorrect. If a series is larger than a converging series, the comparison test does not apply.

The Limit Comparison Test

While the direct comparison test is very useful, there another comparison test that focuses only on the tails of the series that we want to compare. This makes it more widely applicable and simpler to use. We don't need to verify that $a_n \le b_n$ for all (or most) n. However, it will require our skills in evaluating limits at infinity!

THEOREM 14.7 (The Limit Comparison Test). Assume that $a_n > 0$ and $b_n > 0$ for all n (or at least all $n \ge k$) and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

(1) If $0 < L < \infty$ (i.e., L is a positive, *finite* number), then either the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ *both* converge or *both* diverge.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(3) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

The idea of the theorem is since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, then eventually $a_n \approx Lb_n$. So if one of the series converges (diverges) so does the other since the two are 'essentially' scalar multiples of each other.

EXAMPLE 14.19. Does $\sum_{n=1}^{\infty} \frac{1}{3n^2 - n + 6}$ converge?

SOLUTION. SCRAP WORK: Let's apply the limit comparison test. Notice that the terms are always positive since the polynomial $3n^2 - n + 6$ has no roots. In any event, the terms are eventually positive since this an upward-opening parabola. If we focus on highest powers, then the series looks a like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.

ARGUMENT: Since the terms $\frac{1}{3n^2-n+6}$ and $\frac{1}{n^2}$ are positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3n^2-n+6}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2-n+6} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}+\frac{6}{n^2}} = \frac{1}{3} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test ($p = 2 > 1$), then $\sum_{n=1}^{\infty} \frac{1}{3n^2-n+6}$ converges by the limit comparison test (Theorem 14.7).

NOTE: When the series involve fractions, the first step in the limit process can be done more efficiently. Instead of dividing one fraction by the other, we can multiply one fraction by the reciprocal of the other. For instance, earlier in this example we could have written

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{3n^2-n+6} \cdot \frac{n^2}{1}$$

and then carried out the rest of the calculation.

EXAMPLE 14.20 (The General Harmonic Series). The series $\sum_{n=1}^{\infty} \frac{1}{cn+d}$ is called the **general harmonic series**. If $c > 0$ does this series converge?

SOLUTION. SCRAP WORK: Let's apply the limit comparison test by making the obvious comparison to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges.

ARGUMENT: Since the terms $\frac{1}{cn+d}$ are positive once $cn + d > 0$, in other words when $n > -\frac{d}{c}$, and since and $\frac{1}{n}$ is always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{cn+d} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{c+\frac{d}{n}} = \frac{1}{c} > 0,$$

since $c > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test ($p = 1$), then the general harmonic series $\sum_{n=1}^{\infty} \frac{1}{cn+d}$ diverges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.21. Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3+4}$ converge?

SOLUTION. SCRAP WORK: This time if we focus on highest powers, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3+4}$ is roughly equal to $\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$, which diverges

ARGUMENT: Since the terms $\frac{\sqrt{n}}{2n^3+4}$ and $\frac{1}{n^{5/2}}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1/2}}{2n^3+4} \cdot \frac{n^{5/2}}{1} = \lim_{n\to\infty} \frac{n^3}{2n^3+4} = \lim_{n\to\infty} \frac{1}{2+\frac{4}{n^3}} = \frac{1}{2} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges by the p-series test ($p = \frac{5}{2} > 1$), then $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3+4}$ converges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.22. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n+5}}$ converge?

SOLUTION. SCRAP WORK: The obvious comparison is to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which diverges.

ARGUMENT: Since the terms $\frac{1}{\sqrt{4n+5}}$ and $\frac{1}{n^{1/2}}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\sqrt{4n+5}}\cdot\frac{n^{1/2}}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{n^{1/2}}{\sqrt{4n}} = \lim_{n\to\infty}\frac{n^{1/2}}{2n^{1/2}} = \frac{1}{2} > 0.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges by the p-series test ($p = \frac{1}{2} < 1$), then the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{4n+5}}$ diverges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.23. Does the series $\sum_{n=1}^{\infty}\frac{6n^2\cdot 2^n}{n^4+3}$ converge?

SOLUTION. SCRAP WORK: Focusing on highest powers, $\sum_{n=1}^{\infty}\frac{6n^2\cdot 2^n}{n^4+3}$ is roughly $\sum_{n=1}^{\infty}\frac{n^2\cdot 2^n}{n^4} = \sum_{n=1}^{\infty}\frac{2^n}{n^2}$ which we saw is divergent in Example 14.5.

ARGUMENT: Since the terms $\frac{6n^2\cdot 2^n}{n^4+3}$ and $\frac{2^n}{n^2}$ are always positive, we can apply Theorem 14.7. (Note the use of the reciprocal.)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{6n^2\cdot 2^n}{n^4+3}\cdot\frac{n^2}{2^n} = \lim_{n\to\infty}\frac{6n^4}{n^4+3} = \lim_{n\to\infty}\frac{6}{1+\frac{4}{n^3}} = 6 > 0.$$

Since $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ diverges from Example 14.5, then the series $\sum_{n=1}^{\infty}\frac{6n^2\cdot 2^n}{n^4+3}$ diverges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.24. Does the series $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(2\ln(e^n)+2)}{\cos(n\pi)n^3}$ converge?

SOLUTION. SCRAP WORK: First simplify the nth term: $\cos(n\pi) = (-1)^n$ right? And $2\ln(e^n) = 2n$. So $\frac{(-1)^n\cdot(2\ln(e^n)+2)}{\cos(n\pi)n^3} = \frac{2n+2}{n^3}$. Use a limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

ARGUMENT: Since the terms $\frac{(-1)^n\cdot(2\ln(e^n)+2)}{\cos(n\pi)n^3} = \frac{2n+2}{n^3}$ and $\frac{1}{n^2}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+2}{n^3}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{2n^3+2n^2}{n^3} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n^3}{n^3} = 2 > 0.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series test, then the series $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(2\ln(e^n)+2)}{\cos(n\pi)n^3}$ converges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.25. Does the series $\sum_{n=1}^{\infty}\cos\left(\frac{1}{n^2}\right)$ converge?

SOLUTION. SCRAP WORK: The terms $\cos\left(\frac{1}{n^2}\right)$ are always positive. Use a limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

ARGUMENT: Since the terms $\cos\left(\frac{1}{n^2}\right)$ and $\frac{1}{n^2}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\cos\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\cos\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{-\sin\left(\frac{1}{x^2}\right)\cdot\left(-\frac{2}{x^3}\right)}{\left(-\frac{2}{x^3}\right)} = \lim_{x\to\infty} -\sin\left(\frac{1}{x^2}\right) = 0.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series test, then the series $\sum_{n=1}^{\infty}\cos\left(\frac{1}{n^2}\right)$ converges by the second part of the limit comparison test (Theorem 14.7).