The Limit Comparison Test

While the direct comparison test is very useful, there is another comparison test that focuses only on the tails of the series that we want to compare. This makes it more widely applicable and simpler to use. We don't need to verify that $a_n \leq b_n$ for all (or most) n. However, it will require our skills in evaluating limits at infinity!

THEOREM 14.7 (The Limit Comparison Test). Assume that $a_n > 0$ and $b_n > 0$ for all n (or at least all $n \geq k$) and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

(1) If $0 < L < \infty$ (i.e., L is a positive, *finite* number), then either the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ *both* converge or *both* diverge.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(3) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

The idea of the theorem is since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, then eventually $a_n \approx Lb_n$. So if one of the series converges (diverges) so does the other since the two are 'essentially' scalar multiples of each other.

EXAMPLE 14.19. Does $\sum_{n=1}^{\infty} \frac{1}{3n^2 - n + 6}$ converge?

SOLUTION. SCRAP WORK: Let's apply the limit comparison test. Notice that the terms are always positive since the polynomial $3n^2 - n + 6$ has no roots. In any event, the terms are eventually positive since this an upward-opening parabola. If we focus on highest powers, then the series looks a like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.

ARGUMENT: Since the terms $\frac{1}{3n^2-n+6}$ and $\frac{1}{n^2}$ are positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3n^2-n+6}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2 - n + 6} = \lim_{n\to\infty} \frac{1}{3 - \frac{1}{n} + \frac{6}{n^2}} = \frac{1}{3} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test ($p = 2 > 1$), then $\sum_{n=1}^{\infty} \frac{1}{3n^2 - n + 6}$ converges by the limit comparison test (Theorem 14.7).

NOTE: When the series involve fractions, the first step in the limit process can be done more efficiently. Instead of dividing one fraction by the other, we can multiply one fraction by the reciprocal of the other. For instance, earlier in this example we could have written

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{3n^2 - n + 6} \cdot \frac{n^2}{1}$$

and then carried out the rest of the calculation.

EXAMPLE 14.20 (The General Harmonic Series). The series $\sum_{n=1}^{\infty} \frac{1}{cn + d}$ is called the **general harmonic series**. If $c > 0$ does this series converge?

SOLUTION. SCRAP WORK: Let's apply the limit comparison test by making the obvious comparison to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges.

ARGUMENT: Since the terms $\frac{1}{cn+d}$ are positive once $cn + d > 0$, in other words when $n > -\frac{d}{c}$, and since and $\frac{1}{n}$ is always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{cn+d} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{c + \frac{d}{n}} = \frac{1}{c} > 0,$$

since $c > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test ($p = 1$), then the general harmonic series $\sum_{n=1}^{\infty} \frac{1}{cn+d}$ diverges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.21. Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3+4}$ converge?

SOLUTION. SCRAP WORK: This time if we focus on highest powers, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3+4}$ is roughly equal to $\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$, which diverges

ARGUMENT: Since the terms $\frac{\sqrt{n}}{2n^3+4}$ and $\frac{1}{n^{5/2}}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1/2}}{2n^3+4} \cdot \frac{n^{5/2}}{1} = \lim_{n\to\infty} \frac{n^3}{2n^3+4} = \lim_{n\to\infty} \frac{1}{2 + \frac{4}{n^3}} = \frac{1}{2} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges by the p-series test ($p = \frac{5}{2} > 1$), then $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3+4}$ converges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.22. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n+5}}$ converge?

SOLUTION. SCRAP WORK: The obvious comparison is to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which diverges.

ARGUMENT: Since the terms $\frac{1}{\sqrt{4n+5}}$ and $\frac{1}{n^{1/2}}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{4n+5}} \cdot \frac{n^{1/2}}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{4n}} = \lim_{n\to\infty} \frac{n^{1/2}}{2n^{1/2}} = \frac{1}{2} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by the p-series test ($p = \frac{1}{2} < 1$), then the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n+5}}$ diverges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.23. Does the series $\sum_{n=1}^{\infty} \frac{6n^2 \cdot 2^n}{n^4+3}$ converge?

SOLUTION. SCRAP WORK: Focusing on highest powers, $\sum_{n=1}^{\infty} \frac{6n^2 \cdot 2^n}{n^4+3}$ is roughly $\sum_{n=1}^{\infty} \frac{n^2 \cdot 2^n}{n^4} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$ which we saw is divergent in Example 14.5.

ARGUMENT: Since the terms $\frac{6n^2 \cdot 2^n}{n^4+3}$ and $\frac{2^n}{n^2}$ are always positive, we can apply Theorem 14.7. (Note the use of the reciprocal.)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6n^2 \cdot 2^n}{n^4+3} \cdot \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{6n^4}{n^4+3} = \lim_{n\to\infty} \frac{6}{1 + \frac{4}{n^3}} = 6 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges from Example 14.5, then the series $\sum_{n=1}^{\infty} \frac{6n^2 \cdot 2^n}{n^4+3}$ diverges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.24. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3}$ converge?

SOLUTION. SCRAP WORK: First simplify the nth term: $\cos(n\pi) = (-1)^n$ right? And $2\ln(e^n) = 2n$. So $\frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3} = \frac{2n+2}{n^3}$. Use a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

ARGUMENT: Since the terms $\frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3} = \frac{2n+2}{n^3}$ and $\frac{1}{n^2}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+2}{n^3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{2n^3+2n^2}{n^3} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^3}{n^3} = 2 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, then the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3}$ converges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.25. Does the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converge?

SOLUTION. SCRAP WORK: The terms $\sin\left(\frac{1}{n^2}\right)$ are always positive. Use a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

ARGUMENT: Since the terms $\cos\left(\frac{1}{n^2}\right)$ and $\frac{1}{n^2}$ are always positive, we can apply Theorem 14.7.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} &\overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x^2}\right)\cdot\left(-\frac{2}{x^3}\right)}{\left(-\frac{2}{x^3}\right)} \\ &= \lim_{x\to\infty} \cos\left(\frac{1}{x^2}\right) \\ &= \cos 0 = 1.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, then the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ converges by the first part of the limit comparison test (Theorem 14.7).

The Ratio Test

One difficulty with using the two comparison tests is that requires you to know something! It requires you to know something about the convergence or divergence of a similar or related series. The ratio test does not depend on such knowledge—it is a self-contained or self-referential test and the results depend only on the series under consideration. One of its drawbacks, however, is that the test is often inconclusive in terms of deciding whether there is convergence. However, we will see that this test is extremely useful in dealing with so-called power series, which is our final topic of the term. Ok, here's the test.

THEOREM 14.8 (The Ratio Test). Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

Proof. Here's the idea for the proof of part 1. Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. This means that when n is sufficiently large (say $n > k$), we have

$$\frac{a_{n+1}}{a_n} \approx r \Rightarrow a_{n+1} \approx a_n r$$
$$\frac{a_{n+2}}{a_{n+1}} \approx r \Rightarrow a_{n+2} \approx a_{n+1} r \approx a_n r^2$$
$$\frac{a_{n+3}}{a_{n+2}} \approx r \Rightarrow a_{n+3} \approx a_{n+2} r \approx a_n r^3$$
$$\vdots \approx \vdots$$
$$a_{n+k} \approx a_n r^k$$

So the tail of the series is approximately geometric:

$$\sum_{k=0}^{\infty} a_{n+k} \approx \sum_{k=0}^{\infty} a_n r^k.$$

But the geometric series converges since $r < 1$. So $\sum_{n=1}^{\infty} a_n$ converges by the (limit) comparison test. □

Using the Ratio Test

The real utility of this test is that one need not know about another series to determine whether the series under consideration converges. This is very different than with the comparison tests or the integral test where some sort of comparison to another series is required.

EXAMPLE 14.26. Determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

SOLUTION. This is not a geometric series, but the terms are positive so let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2} < 1.$$

By the ratio test the series converges.

That was quick! The ratio test is very easy to use with both factorials and exponentials (powers) because there will be a lot of cancelation.

EXAMPLE 14.27. Determine whether $\sum_{n=1}^{\infty} \frac{3^{n+2} n!}{4^n}$ converges.

SOLUTION. There are both exponentials and factorials and the terms are positive, so let's try the ratio test. To eliminate compound fractions we can simplify the limit expression by multiplying by the reciprocal of a_n instead of dividing by it.
ARGUMENT:

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+2}(n+1)!}{4^{n+1}} \cdot \frac{4^n}{3^{n+1} n!} = \lim_{n\to\infty} \frac{3(n+1)}{4} = \infty.$$

By the ratio test the series diverges. That was quick!

EXAMPLE 14.28. Here's a very similar one: Determine whether $\sum_{n=1}^{\infty} \frac{8^{n+1} n^2}{2^{2n}}$ converges.

SOLUTION. Because of the exponentials let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{8^{n+2}(n+1)^2}{2^{2n+2}} \cdot \frac{2^n}{8^{n+1}n^2} = \lim_{n\to\infty} \frac{8(n+1)^2}{2^2 n^2} = \lim_{n\to\infty} \frac{2(n^2+2n+1)}{n^2}$$
$$\overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^2}{n^2} = 2 > 1.$$

By the ratio test the series diverges.

EXAMPLE 14.29. Determine whether $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges.

SOLUTION. Because of the exponentials let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n} = \lim_{n\to\infty} \frac{3n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{3}{n+1} \cdot \frac{n^n}{(n+1)^n}$$
$$= \lim_{n\to\infty} \frac{3}{n+1} \left(\frac{n}{n+1}\right)^n$$
$$= \lim_{n\to\infty} \frac{3}{n+1} \left(\frac{1}{1+\frac{1}{n}}\right)^n$$
$$= \lim_{n\to\infty} 0 \cdot \frac{1}{e} = 0$$

By the ratio test the series converges.

Why is the test inconclusive when the ratio is 1? The next example shows why.

EXAMPLE 14.30. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the p-series test we know that the former diverges while the later converges. But notice what happens when we try to apply the ratio test to each.

With the harmonic series we find

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{n} = 1.$$

With the second series we also get

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^2}{n^2} = 1.$$

Both ratios are 1, yet the first series diverges and the other converges (p-series test). For this reason the ratio test is inconclusive when the limit is 1.

Let's look at a few more examples.

EXAMPLE 14.31. Determine whether $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$ converges.

SOLUTION. Because of the exponential and factorial let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n} = \lim_{n\to\infty} \frac{2}{(2n+1)(2n+2)} = 0 < 1.$$

By the ratio test the series converges.

EXAMPLE 14.32. Here's a slightly more complicated one: Determine whether $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converges.

SOLUTION. Because of the factorials let's try the ratio test.
ARGUMENT: The terms are positive and

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+2)!}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!} = \lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{(2n+1)2}{(n+1)}$$
$$\overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{4n}{n} = 4 > 1.$$

By the ratio test the series diverges.

The Root Test

The root is similar to the ratio test, though a bit less useful. It is a good test to use with series that contain powers but not so useful for series with factorials. The set up is essentially the same.

THEOREM 14.9 (The Root Test). Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty}\sqrt[n]{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty}\sqrt[n]{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

Proof. Here's the rough idea. Suppose that $\lim_{n\to\infty}\sqrt[n]{a_n} = r$. This means that when n is sufficiently large (say $n > k$), we have $a^n \approx r^n$ So the tail of the series is approximately geometric:

$$\sum_{k=n}^{\infty} a_k \approx \sum_{k=n}^{\infty} r^k.$$

But a geometric series converges if and only if $r < 1$. If the tail of the series converges, so does the entire series. So the entire series $\sum_{n=1}^{\infty} a_n$ will converge if $r < 1$ and certainly diverge if $r > 1$. When $r = 1$, the test turns out to be inconclusive, as we will see. □

Using the Root Test

The real utility of this test is that one need not know about another series to determine whether the series under consideration converges. Again, this is very different than with the comparison tests where some knowledge of another series is required.

EXAMPLE 14.33. Determine whether $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges.

SOLUTION. This is not a geometric series, but the terms are positive and involve powers, so let's try the root test. (We just did this by the ratio test. Compare the arguments.)
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{1}{2},$$

using a key limit. By the root test the series converges. That was still pretty easy.

That was quick! The root test is very easy to use with exponentials (powers) because there will be a lot of cancelation.

EXAMPLE 14.34. Here's a another we did with the ratio test: Determine whether $\sum_{n=1}^{\infty} \frac{8^{n+1}n^2}{2^{2n}}$ converges.

SOLUTION. Because of the powers let's try the root test.
ARGUMENT: The terms are positive and

$$\begin{aligned} r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{8^{n+1}n^2}{2^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{8^{n+1}n^2}}{\sqrt[n]{2^{2n}}} &= \lim_{n\to\infty} \frac{8^{(n+1)/n}n^{2/n}}{2^{2n/n}} \\ &= \lim_{n\to\infty} \frac{8^{1+1/n}(n^{1/n})^2}{2^2} \\ &= \lim_{n\to\infty} \frac{8\cdot 1}{4} \\ &= 2 > 1. \end{aligned}$$

By the root test the series diverges. Was this a bit more of a pain than doing it by the ratio test?

Why is the test inconclusive when the root is 1? The next example shows why.

EXAMPLE 14.35. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the p-series test we know that the former diverges while the later converges. But notice what happens when we try to apply the root test to each.

With the harmonic series we find

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{\sqrt[n]{1}}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

With the second series we also get

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \frac{\sqrt[n]{1}}{[\sqrt[n]{n}]^2} = \frac{1}{1^2} = 1.$$

Both roots are 1, yet the first series diverges and the other converges (p-series test). For this reason the root test is inconclusive when the limit is 1.

EXAMPLE 14.36. Determine whether $\sum_{n=1}^{\infty} \left(\frac{2n^3+1}{6n^3+n+2}\right)^{3n}$ converges.

SOLUTION. Because of the power let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+1}{6n^3+n+2}\right)^{3n}} = \lim_{n\to\infty} \left(\frac{2n^3+1}{6n^3+n+2}\right)^3 \overset{\text{HPwrs}}{=} \left(\frac{1}{3}\right)^3 < 1.$$

By the root test the series converges.

EXAMPLE 14.37. Determine whether $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^{n^2}$ converges.

SOLUTION. Because of the power let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{2}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n^2/n} = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n} = e^2 > 1.$$

By the root test the series diverges.