EXAMPLE 14.24. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3}$ converge?

SOLUTION. SCRAP WORK: First simplify the nth term: $\cos(n\pi) = (-1)^n$ right? And $2\ln(e^n) = 2n$. So $\frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3} = \frac{2n+2}{n^3}$. Use a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
ARGUMENT: Since the terms $\frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3} = \frac{2n+2}{n^3}$ and $\frac{1}{n^2}$ are always positive, we can apply Theorem 14.7.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+2}{n^3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{2n^3+2n^2}{n^3} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^3}{n^3} = 2 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, then the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2\ln(e^n)+2)}{\cos(n\pi)n^3}$ converges by the limit comparison test (Theorem 14.7).

EXAMPLE 14.25. Does the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converge?

SOLUTION. SCRAP WORK: The terms $\sin\left(\frac{1}{n^2}\right)$ are always positive. Use a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
ARGUMENT: Since the terms $\cos\left(\frac{1}{n^2}\right)$ and $\frac{1}{n^2}$ are always positive, we can apply Theorem 14.7.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} &\overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x^2}\right) \cdot \left(-\frac{2}{x^3}\right)}{\left(-\frac{2}{x^3}\right)} \\ &= \lim_{x\to\infty} \cos\left(\frac{1}{x^2}\right) \\ &= \cos 0 = 1.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, then the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ converges by the first part of the limit comparison test (Theorem 14.7).

The Ratio Test

One difficulty with using the two comparison tests is that requires you to know something! It requires you to know something about the convergence or divergence of a similar or related series. The ratio test does not depend on such knowledge—it is a self-contained or self-referential test and the results depend only on the series under consideration. One of its drawbacks, however, is that the test is often inconclusive in terms of deciding whether there is convergence. However, we will see that this test is extremely useful in dealing with so-called power series, which is our final topic of the term. Ok, here's the test.

THEOREM 14.8 (The Ratio Test). Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

Proof. Here's the idea for the proof of part 1. Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. This means that when n is sufficiently large (say $n > k$), we have

$$\begin{aligned} \frac{a_{n+1}}{a_n} &\approx r \Rightarrow a_{n+1} \approx a_n r \\ \frac{a_{n+2}}{a_{n+1}} &\approx r \Rightarrow a_{n+2} \approx a_{n+1} r \approx a_n r^2 \\ \frac{a_{n+3}}{a_{n+2}} &\approx r \Rightarrow a_{n+3} \approx a_{n+2} r \approx a_n r^3 \\ \vdots &\approx \vdots \\ a_{n+k} &\approx a_n r^k \end{aligned}$$

So the tail of the series is approximately geometric:

$$\sum_{k=0}^{\infty} a_{n+k} \approx \sum_{k=0}^{\infty} a_n r^k.$$

But the geometric series converges since $r < 1$. So $\sum_{n=1}^{\infty} a_n$ converges by the (limit) comparison test. □

Using the Ratio Test

The real utility of this test is that one need not know about another series to determine whether the series under consideration converges. This is very different than with the comparison tests or the integral test where some sort of comparison to another series is required.

EXAMPLE 14.26. Determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

SOLUTION. This is not a geometric series, but the terms are positive so let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2} < 1.$$

By the ratio test the series converges.

That was quick! The ratio test is very easy to use with both factorials and exponentials (powers) because there will be a lot of cancelation.

EXAMPLE 14.27. Determine whether $\sum_{n=1}^{\infty} \frac{3^{n+2} n!}{4^n}$ converges.

SOLUTION. There are both exponentials and factorials and the terms are positive, so let's try the ratio test. To eliminate compound fractions we can simplify the limit expression by multiplying by the reciprocal of a_n instead of dividing by it.
ARGUMENT:

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+2}(n+1)!}{4^{n+1}} \cdot \frac{4^n}{3^{n+1} n!} = \lim_{n\to\infty} \frac{3(n+1)}{4} = \infty.$$

By the ratio test the series diverges. That was quick!

EXAMPLE 14.28. Here's a very similar one: Determine whether $\sum_{n=1}^{\infty} \frac{8^{n+1} n^2}{2^{2n}}$ converges.

SOLUTION. Because of the exponentials let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{8^{n+2}(n+1)^2}{2^{2n+2}} \cdot \frac{2^n}{8^{n+1}n^2} = \lim_{n\to\infty} \frac{8(n+1)^2}{2^2n^2} = \lim_{n\to\infty} \frac{2(n^2+2n+1)}{n^2} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{2n^2}{n^2} = 2 > 1.$$

By the ratio test the series diverges.

EXAMPLE 14.29. Determine whether $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges.

SOLUTION. Because of the exponentials let's try the ratio test.
ARGUMENT: The terms are positive and

$$\begin{aligned} r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n} = \lim_{n\to\infty} \frac{3n^n}{(n+1)^{n+1}} &= \lim_{n\to\infty} \frac{3}{n+1} \cdot \frac{n^n}{(n+1)^n} \\ &= \lim_{n\to\infty} \frac{3}{n+1} \left(\frac{n}{n+1}\right)^n \\ &= \lim_{n\to\infty} \frac{3}{n+1} \left(\frac{1}{1+\frac{1}{n}}\right)^n \\ &= \lim_{n\to\infty} 0 \cdot \frac{1}{e} = 0 \end{aligned}$$

By the ratio test the series converges.

Why is the test inconclusive when the ratio is 1? The next example shows why.

EXAMPLE 14.30. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the p-series test we know that the former diverges while the later converges. But notice what happens when we try to apply the ratio test to each.

With the harmonic series we find

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n}{n} = 1.$$

With the second series we also get

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{n^2}{n^2} = 1.$$

Both ratios are 1, yet the first series diverges and the other converges (p-series test). For this reason the ratio test is inconclusive when the limit is 1.

Let's look at a few more examples.

EXAMPLE 14.31. Determine whether $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$ converges.

SOLUTION. Because of the exponential and factorial let's try the ratio test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n} = \lim_{n\to\infty} \frac{2}{(2n+1)(2n+2)} = 0 < 1.$$

By the ratio test the series converges.

EXAMPLE 14.32. Here's a slightly more complicated one: Determine whether $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converges.

SOLUTION. Because of the factorials let's try the ratio test.
ARGUMENT: The terms are positive and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{(2n+1)2}{(n+1)} \stackrel{\text{HPwrs}}{=} \lim_{n\to\infty} \frac{4n}{n} = 4 > 1.$$

By the ratio test the series diverges.

The Root Test

The root is similar to the ratio test, though a bit less useful. It is a good test to use with series that contain powers but not so useful for series with factorials. The set up is essentially the same.

THEOREM 14.9 (The Root Test). Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \sqrt[n]{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

Proof. Here's the rough idea. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = r$. This means that when n is sufficiently large (say $n > k$), we have $a^n \approx r^n$ So the tail of the series is approximately geometric:

$$\sum_{k=n}^{\infty} a_k \approx \sum_{k=n}^{\infty} r^k.$$

But a geometric series converges if and only if $r < 1$. If the tail of the series converges, so does the entire series. So the entire series $\sum_{n=1}^{\infty} a_n$ will converge if $r < 1$ and certainly diverge if $r > 1$. When $r = 1$, the test turns out to be inconclusive, as we will see. □

Using the Root Test

The real utility of this test is that one need not know about another series to determine whether the series under consideration converges. Again, this is very different than with the comparison tests where some knowledge of another series is required.

EXAMPLE 14.33. Determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

SOLUTION. This is not a geometric series, but the terms are positive and involve powers, so let's try the root test. (We just did this by the ratio test. Compare the arguments.)
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{1}{2},$$

using a key limit. By the root test the series converges. That was still pretty easy.

That was quick! The root test is very easy to use with exponentials (powers) because there will be a lot of cancelation.

EXAMPLE 14.34. Here's a another we did with the ratio test: Determine whether $\sum_{n=1}^{\infty} \frac{8^{n+1}n^2}{2^{2n}}$ converges.

SOLUTION. Because of the powers let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{8^{n+1}n^2}{2^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{8^{n+1}n^2}}{\sqrt[n]{2^{2n}}} = \lim_{n\to\infty} \frac{8^{(n+1)/n}n^{2/n}}{2^{2n/n}}$$
$$= \lim_{n\to\infty} \frac{8^{1+1/n}(n^{1/n})^2}{2^2}$$
$$= \lim_{n\to\infty} \frac{8\cdot 1}{4}$$
$$= 2 > 1.$$

By the root test the series diverges. Was this a bit more of a pain than doing it by the ratio test?

Why is the test inconclusive when the root is 1? The next example shows why.

EXAMPLE 14.35. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the p-series test we know that the former diverges while the later converges. But notice what happens when we try to apply the root test to each.

With the harmonic series we find

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \frac{\sqrt[n]{1}}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

With the second series we also get

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \frac{\sqrt[n]{1}}{[\sqrt[n]{n}]^2} = \frac{1}{1^2} = 1.$$

Both roots are 1, yet the first series diverges and the other converges (p-series test). For this reason the root test is inconclusive when the limit is 1.

EXAMPLE 14.36. Determine whether $\sum_{n=1}^{\infty} \left(\frac{2n^3+1}{6n^3+n+2}\right)^{3n}$ converges.

SOLUTION. Because of the power let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+1}{6n^3+n+2}\right)^{3n}} = \lim_{n\to\infty} \left(\frac{2n^3+1}{6n^3+n+2}\right)^3 \stackrel{\text{HPwrs}}{=} \left(\frac{1}{3}\right)^3 < 1.$$

By the root test the series converges.

EXAMPLE 14.37. Determine whether $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^{n^2}$ converges.

SOLUTION. Because of the power let's try the root test.
ARGUMENT: The terms are positive and

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{2}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n^2/n} = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n} = e^2 > 1.$$

By the root test the series diverges.