8.4 *Trig Integrals*

This section is devoted to integrating powers of trig functions. First we examine powers of sine and cosine functions.

Powers of a Single Trig Function

We begin with four key trig identities that you should **memorize** that will make your life and these integrals much simpler.

Four Key Identities.

$\cos^2 u + \sin^2 u = 1$	(so $\sin^2 u = 1 - \cos^2 u$ or $\cos^2 u = 1 - \sin^2 u$)
$1 + \tan^2 u = \sec^2 u$	(so $\tan^2 u = \sec^2 u - 1$)
$\sin^2 u = \frac{1}{2} - \frac{1}{2}\cos 2u$	(Half angle formula).
$\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$	

The half angle formulas are used to integrate $\sin^2 u$ or $\cos^2 u$ in the obvious way.

EXAMPLE 8.4.1. Determine $\int \cos^2(8x)\,dx$.

SOLUTION. Use equation (3) above with $u = 8x$. Note the use of a 'mental adjustment.'

$$\int \cos^2(8x)\,dx = \int \tfrac{1}{2} + \tfrac{1}{2}\cos(16x)\,dx = \tfrac{1}{2}x + \tfrac{1}{32}\sin(16x) + c.$$

We have already seen how to integrate low powers of the secant and tangent functions.

$\int \tan u\,du = \ln\lvert\sec u\rvert + c$	
$\int \tan^2 u\,du = \int \sec^2 u - 1\,du = \tan u - u + c$	(Use a key trig identity.)
$\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert + c$	
$\int \sec^2 u\,du = \tan u + c$	

Higher Powers of Trig Functions

When we want to integrate higher powers (powers greater than $n \geq 2$) of a single trig function we can make use of integration by parts.

EXAMPLE 8.4.2. Suppose that $n \geq 2$ is an integer. Determine $\int \sin^n x\,dx$.

SOLUTION. Use integration by parts. The key is to write $\sin^n x$ as $\sin^{n-1} x \sin x$.

$u = \sin^{n-1} x$	$dv = \sin x\,dx$
$du = (n-1)\sin^{n-2} x \cos x\,dx$	$v = -\cos x$

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\left[\int \sin^{n-2} x - \sin^n x\,dx\right]\end{aligned}$$

Combining all the $\sin^n x$ terms,

$$n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx$$

So we obtain

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

This is what is known as a reduction formula, and it can be used repeatedly to determine integrals of high powers of $\sin x$.

EXAMPLE 8.4.3. Determine $\int \sin^5 x\,dx$.

SOLUTION. Use the reduction formula with $n = 5$.

$$\int \sin^5 x\,dx = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\,dx.$$

Now use it again with $n = 3$.

$$\int \sin^5 x\,dx = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\,dx\right].$$

Now we can finish the integration in the usual way.

$$\int \sin^5 x\,dx = -\frac{\sin^4 x \cos x}{5} - \frac{4\sin^2 x \cos x}{15} - \frac{8\cos x}{15} + c.$$

Reduction Formulas for Large Powers. There are reduction formulas for the other trig functions as well. They are verified using integration by parts. The most important are (these do not need to be memorized). Repeated application may be necessary.

$$\int \cos^n u\,du = \tfrac{1}{n}\cos^{n-1} u \sin u + \tfrac{n-1}{n}\int \cos^{n-2} u\,du$$

$$\int \sin^n u\,du = -\tfrac{1}{n}\sin^{n-1} u \cos u + \tfrac{n-1}{n}\int \sin^{n-2} u\,du$$

$$\int \tan^n u\,du = \tfrac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$$

$$\int \sec^n u\,du = \tfrac{1}{n-1}\sec^{n-2} u \tan u + \tfrac{n-2}{n-1}\int \sec^{n-2} u\,du$$

EXAMPLE 8.4.4. Determine $\int \tan^4 x\,dx$.

SOLUTION. Use the reduction formula with $n = 4$.

$$\begin{aligned}\int \tan^4 x\,dx &= \frac{\tan^3 x}{3} - \int \tan^2 x\,dx \\ &= \frac{\tan^3 x}{3} - \left[\frac{\tan x}{1} - \int 1\,dx\right] \\ &= \frac{\tan^3 x}{3} - \tan x + x + c.\end{aligned}$$

Notice that we used the fact that $\tan^0 x = 1$.

Integrating Products of Powers of Sines and Cosines

We will now develop some guidelines for integrals of the form

$$\int \sin^m x \cos^n x\,dx$$

where either m or n is a positive integer. Notice that both trig functions have the same argument. The goal is to use the power rule, as in the following: Determine $\int \sin x \cos^4 x\,dx$. You should recognize this as a simple substitution problem; let $u = \cos x$ and $du = -\sin x\,dx$. Then

$$\int \sin x \cos^4 x\,dx = -\int u^4\,du = -\frac{u^5}{5} + c = -\frac{\cos^5 x}{5} + c.$$

Thus, the protocol is to use trig identities and u-substitution to turn the integral into a simple power rule problem.

Guidelines for Products of Powers of Sines and Cosines: These general principles can help you solve integrals of the form $\int \sin^m x \cos^n x \, dx$.

1. If the power of sine is odd and positive, split off a factor of sine for du and convert the rest to cosines, let $u = \cos x$, and then integrate. For example,

$$\int \overbrace{\sin^{2k+1} x}^{m=2k+1 \text{ odd}} \cos^n x \, dx = \int (\sin^2 x)^k x \cos^n x \cdot \overbrace{\sin x \, dx}^{\text{use for } du}$$
$$= \int \overbrace{(1-\cos^2 x)^k}^{\text{convert to cosines}} \cos^n x \cdot \sin x \, dx = -\int (1-u^2)^k u^n \, du.$$

2. If the power of cosine is odd and positive (and the power of sine is even), split off a factor of cosine for du and convert the rest to sines, let $u = \sin x$, and then integrate. For example,

$$\int \sin^m x \overbrace{\cos^{2k+1} x}^{n=2k+1 \text{ odd}} \, dx = \int \sin^m x (\cos^2 x)^k \cdot \overbrace{\cos x \, dx}^{\text{use for } du}$$
$$= \int \sin^m x \overbrace{(1-\sin^2 x)^k}^{\text{convert to sines}} \cdot \cos x \, dx = \int u^m (1-u^2)^k \, du.$$

3. If both powers of sine and cosine are *even* and non-negative, make repeated use of the identities $\sin^2 u = \frac{1}{2} - \frac{1}{2}\cos 2u$ and $\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$ to powers of cosines. Then use reduction formula #1.

4. Use a table of integrals or *WolframAlpha* or other software. Certainly you should use this tool in later courses whether in math or other departments.

EXAMPLE 8.4.5 (Using Rule 1). Determine $\int \sin^3 x \cos^2 x \, dx$.

SOLUTION. Since the power of the sine function is odd, we use Guideline #1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \cdot \sin x \, dx && \text{split off a power of } \sin x \\
&= \int (1-\cos^2 x)\cos^2 x \cdot \sin x \, dx && \text{use a trig id} \\
&= -\int (1-u^2)u^2 \, du && \text{substitute } u = \cos x,\ du = -\sin x \\
&= -\int u^2 + u^4 \, du && \text{expand} \\
&= -\frac{u^3}{3} + \frac{u^5}{5} + c \\
&= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c
\end{aligned}$$

EXAMPLE 8.4.6 (Using Rule 2). Determine $\int \sin^4 3x \cos^3 3x \, dx$.

SOLUTION. Both functions have the same argument (angle). The sine function has an even power and the power of the cosine function is odd; we use rule #2.

$$\begin{aligned}
\int \sin^4 3x \cos^3 3x\,dx &= \int \sin^4 3x \cos^2 3x \cdot \cos 3x\,dx && \text{split off a power of } \cos 3x \\
&= \int \sin^4 3x(1-\sin^2 3x)\cos 3x\,dx && \text{use a trig id} \\
&= \frac{1}{3}\int u^4(1-u^2)\,du && u=\sin 3x,\ \tfrac{1}{3}du = \cos 3x\,dx \\
&= \frac{1}{3}\int u^4 - u^6\,du && \text{expand} \\
&= \frac{u^5}{15} - \frac{u^7}{21} + c \\
&= \frac{\sin^5 3x}{15} - \frac{\sin^7 3x}{21} + c
\end{aligned}$$

Gosh, be careful of the mental adjustment required for the substitution.

EXAMPLE 8.4.7 (Using Rule 3). Determine $\int \sin^2 5x \cos^2 5x\,dx$.

SOLUTION. The powers of the sine and cosine function are even, we use Guideline #3.

$$\begin{aligned}
\int \sin^2 5x \cos^2 5x\,dx &= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 10x\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 10x\right)\,dx && \text{half angle formula} \\
&= \int \tfrac{1}{4} - \tfrac{1}{4}\cos^2 10x\,dx && \text{expand} \\
&= \int \tfrac{1}{4} - \tfrac{1}{4}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 20x\right]\,dx && \text{half angle formula} \\
&= \int \tfrac{1}{8} - \tfrac{1}{8}\cos 20x\,dx && \text{combine terms} \\
&= \tfrac{1}{8}x - \tfrac{1}{160}\sin 20x + c && \text{mental adjustment}
\end{aligned}$$

Again not too bad! Just be careful.

EXAMPLE 8.4.8 (Using Rule 2). Determine $\int \cos^5 x\,dx$.

SOLUTION. Since the sine function does not appear and the power of the cosine function is odd, we can use Guideline #2. A preferred (?) method might be to use a reduction formula.

$$\begin{aligned}
\int \cos^5 x\,dx &= \int (\cos^2 x)^2 \cos x\,dx && \text{split off a power of } \cos x \\
&= \int (1-\sin^2 x)^2 \cos x\,dx && \text{use a trig id} \\
&= \int (1-u^2)^2\,du && u=\sin x,\ du=\cos x\,dx \\
&= \int 1 - 2u^2 + u^4\,du && \text{expand} \\
&= u - \frac{2u^3}{3} + \frac{u^5}{5} + c \\
&= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c
\end{aligned}$$

Not too bad!

Integrating Products of Powers of Secant and Tangent

Here's some material from your text that we will use to integrate powers of tangent and secant. You do not need to memorize this. But you do need to know how to use this information. The idea is very similar to what we used for products of sines and cosines.

Guidelines for Products of Powers of Tangents and Secants: These general principles can help you solve integrals of the form $\int \tan^m x \sec^n x\, dx$.

1. If the power of secant is *even* and positive, split off $\sec^2 x$ to use for du and convert the rest to tangents, then let $u = \tan x$, and integrate. For example,

$$\int \tan^m x \overbrace{\sec^{2k} x}^{n=2k \text{ even}}\, dx = \int \tan^m x (\sec^2 x)^{k-1} \cdot \overbrace{\sec^2 x\, dx}^{\text{use for } du}$$
$$= \int \tan^m x \overbrace{(1+\tan^2 x)^{k-1}}^{\text{convert to tangents}} \cdot \sec^2 x\, dx = \int u^m (1+u^2)^{k-1}\, du.$$

2. If the power of tangent is odd and positive (and the power of secant is odd), split off $\sec x \tan x$ for du and convert the rest to secants, let $u = \sec x$, and then integrate. For example,

$$\int \overbrace{\tan^{2k+1} x}^{m=2k+1 \text{ odd}} \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x \cdot \overbrace{\sec x \tan x\, dx}^{\text{use for } du}$$
$$= \int \overbrace{(\sec^2 x - 1)^k}^{\text{convert to secants}} \sec^{n-1} x \cdot \sec x \tan x\, dx = \int (u^2-1)^k u^{n-1}\, du.$$

3. If m is even and n is odd, convert the tangents to secants and use the reduction formula for powers of the secant:

$$\int \overbrace{\tan^{2k} x}^{m=2k \text{ even}} \overbrace{\sec^n x}^{n \text{ odd}}\, dx = \int \overbrace{(\sec^2 x - 1)^k}^{\text{convert to secants}} \sec^n x\, dx.$$

4. In real life, use *WolframAlpha*, or look in a table of integrals.

EXAMPLE 8.4.9 (Using Guideline 2). Determine $\displaystyle\int \tan^3 x \sec^3 x\, dx$.

SOLUTION. The powers of both secant and tangent are odd, so Guideline #2 above applies.

$$\begin{aligned}
\int \tan^3 x \sec^3 x\, dx &= \int \tan^2 x \sec^2 x (\sec x \tan x)\, dx && \text{split off a secant-tangent} \\
&= \int (\sec^2 x - 1)\sec^2 x (\sec x \tan x)\, dx && \text{convert to secants} \\
&= \int (u^2-1)u^2\, du && u = \sec x, \quad du = \sec x \tan x \\
&= \int u^4 - u^2\, du && \text{expand} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + c \\
&= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + c
\end{aligned}$$

EXAMPLE 8.4.10 (Using Guideline 1). Determine $\displaystyle\int \tan^4 x \sec^4 x\, dx$.

SOLUTION. The powers of both secant and tangent are even, so Guideline #1 applies.

$$\begin{aligned}
\int \tan^4 x \sec^4 x\,dx &= \int \tan^4 x \sec^2 x \cdot \sec^2 x\,dx && \text{split off a secant-squared} \\
&= \int \tan^4 x (1+\tan^2 x) \cdot \sec^2 x\,dx && \text{convert to tangents} \\
&= \int u^4 (1+u^2)\,du && u = \tan x, \quad du = \sec^2 x \\
&= \int u^4 + u^6\,du && \text{expand} \\
&= \frac{u^5}{5} + \frac{u^7}{7} + c \\
&= \tfrac{1}{5}\tan^5 x + \tfrac{1}{7}\tan^7 x + c
\end{aligned}$$

EXAMPLE 8.4.11 (Using Guideline 3). Determine $\int \tan^2 x \sec x\,dx$.

SOLUTION. The powers of tangent is even and the power of secant is odd, so Guideline #3 applies.

$$\begin{aligned}
\int \tan^2 x \sec x\,dx &= \int (\sec^2 x - 1) \sec x\,dx && \text{convert to secants} \\
&= \int \sec^3 x\,dx - \int \sec x\,dx && \text{expand} \\
&= \left[\frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx\right] - \int \sec x\,dx && \text{reduction formula} \\
&= \frac{\sec x \tan x}{2} - \frac{1}{2}\int \sec x\,dx && \text{combine, simplify} \\
&= \frac{\sec x \tan x}{2} - \frac{1}{2}\ln|\sec x + \tan x| + c.
\end{aligned}$$

Take-Home Message. OK, the goal here is to be able to use these guidelines for products of trig functions. Here's what you must be able to do without looking back to the guidelines:

1. You need to KNOW the four key identities.
2. You need to be able to integrate low powers ($n = 1$ or 2) of the four main trig functions.
3. You need to be able to integrate products of the form $\sin^n x \cos^m x$ when at least one of the powers is an odd positive integer by splitting off an appropriate factor, using a key trig id, and then using u-substitution (Guidelines 1 and 2 for Products of Sines and Cosines).
4. You do NOT need to memorize the reduction formulæ, do not need to memorize the guidelines for products of the form $\tan^m x \sec^n x$.