Integration by Triangle Substitutions

The Area of a Circle

So far we have used the technique of u-substitution (i.e., reversing the chain rule) and integration by parts (reversing the product rule) to extend the “list" of functions that we can antidifferentiate. Remember that we are in the antidifferentiation ‘business’ because the Fundamental Theorem (FTC) says that if F is an antiderivative of f, then the area under f on the interval $[a, b]$ is $\int_a^b f(x)\,dx = F(b) - F(a)$. This theorem “solves" the area problem, at least for those functions whose antiderivatives we know. But even with our new integration techniques there are many integrals we cannot yet do, such as:

EXAMPLE 7.2.1. Find the area of the semi-circle of radius r. Since middle school we have been told that the area of a circle is πr^2, so the area of a semi-circle is $\frac{1}{2}\pi r^2$. But why is this formula valid? Recall that the entire circle of radius r centered at the origin is the set of points that satisfies $x^2 + y^2 = r^2$. It follows that the upper semi-circle is given by the function $y = f(x) = \sqrt{r^2 - x^2}$ on the interval $[-r, r]$. So the area we seek is

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx.$$

It’s scandalous but true: none of the techniques we’ve developed so far will help us find an appropriate antiderivative. Still $\sqrt{r^2 - x^2}$ should remind us of right triangles, and we’ve excised one such triangle from the figure below.

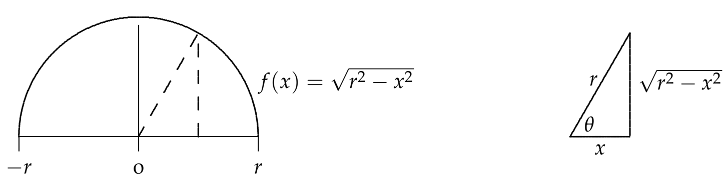


Figure 7.1: Left: A semi-circle with equation $y = \sqrt{r^2 - x^2}$. Right: A corresponding right triangle.

The sides of the triangle are related to each other via the trigonometric functions of the angle θ. We can use these to set up a fancy u-substitution, or more appropriately, a θ-substitution.

$$\begin{aligned}
\frac{\sqrt{r^2 - x^2}}{r} = \sin\theta \Rightarrow &\qquad \sqrt{r^2 - x^2} = r\sin\theta \\
\frac{x}{r} = \cos\theta \Rightarrow &\qquad x = r\cos\theta \\
\Rightarrow &\qquad dx = -r\sin\theta\,d\theta
\end{aligned}$$

Notice that each part of the original integrand can now be written in terms of the angle θ. Of course, we need to write the x-limits of integration as θ-limits. When

$$\begin{aligned}
x = -r = r\cos\theta \quad &\Rightarrow \quad \cos\theta = -1 \quad \Rightarrow \quad \theta = \pi \\
x = r = r\cos\theta \quad &\Rightarrow \quad \cos\theta = 1 \quad \Rightarrow \quad \theta = 0
\end{aligned}$$

This makes sense: θ changes from π to 0 as you move around the semi-circle clockwise from $-r$ to r. Now the integral can be rewritten in terms of θ and solved.

$$\begin{aligned}
\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \int_{\pi}^{0} r\sin\theta(-r\sin\theta)\, d\theta = -r^2 \int_{\pi}^{0} \sin^2\theta\, d\theta &= -r^2 \int_{\pi}^{0} \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\, d\theta \\
&= -r^2 \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta)\right)\Big|_{\pi}^{0} \\
&= -r^2 \left[(0-0) - \left(\frac{\pi}{2} - 0\right)\right] \\
&= \frac{\pi r^2}{2}
\end{aligned}$$

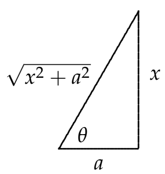
From this it follows that **the area of a circle of radius r is** πr^2.

7.3 *General Triangle Substitutions*

In general, triangle substitutions are based on, well, triangles! Right triangles, in particular. Three different substitutions arise depending on the integrand and how the sides of the triangle are labeled. The triangles are based on the corresponding forms of the square roots that arise from the Pythagorean theorem in these triangles. The three cases are: $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$, and $\sqrt{a^2-x^2}$.

Case 1: Integrals involving $\sqrt{x^2+a^2}$

In this case, $\sqrt{a^2+x^2}$ must correspond to the hypotenuse of the right triangle. (Why?) So with the legs of the triangle labeled as below, we have:



$$x = a\tan\theta$$

$$dx = a\sec^2\theta\, d\theta$$

$$\sqrt{x^2+a^2} = a\sec\theta$$

EXAMPLE 7.3.1. Calculate the following indefinite integral: $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\, dx$.

Using the triangle substitution in the box above with $a^2 = 4$ or $a = 2$, we have:

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\, dx = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta\, d\theta = \tfrac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta &= \tfrac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta \\
&= \tfrac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \\
&= \tfrac{1}{4}\int (\sin\theta)^{-2}\cos\theta\, d\theta \\
&= -\tfrac{1}{4}(\sin\theta)^{-1} + c
\end{aligned}$$

We must convert our answer back to a function of x. Look back at the triangle above. Notice that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$. So

$$-\tfrac{1}{4}(\sin\theta)^{-1} + c = -\tfrac{1}{4}\left(\frac{x}{\sqrt{x^2+4}}\right)^{-1} = -\frac{\sqrt{x^2+4}}{4x} + c.$$

One question you may have is how should you select which leg of the triangle should correspond to x and which to a? The answer is that the substitution will work no matter which you use, but it will be easier with x as the opposite side to angle θ. If we had let x be the adjacent side, then $x = a\cot\theta$, a less familiar trig function.

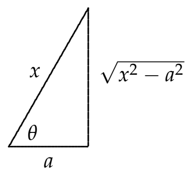
EXAMPLE 7.3.2. Calculate the indefinite integral: $\displaystyle\int \frac{1}{\sqrt{16+x^2}}\, dx$.

Using the triangle substitution above with $a^2 = 16$ or $a = 4$, we have:

$$\int \frac{1}{\sqrt{16+x^2}}\,dx = \int \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + c = \ln\left|\tfrac{\sqrt{16+x^2}}{4} + \tfrac{x}{4}\right| + c$$

Case 2: Integrals involving $\sqrt{x^2 - a^2}$

In this case, x must correspond to the hypotenuse of the right triangle. (Why?) Again we have our choice of how to label the legs, one side a and the other $\sqrt{x^2 - a^2}$. With the selection made below, $x = a\sec\theta$. What would x equal if we had let a be the side opposite θ?



$$x = a\sec\theta$$
$$dx = a\sec\theta\tan\theta\,d\theta$$
$$\sqrt{x^2 - a^2} = a\tan\theta$$

EXAMPLE 7.3.3. Evaluate the definite integral: $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$.

Use the triangle substitution above with $a^2 = 1$ or $a = 1$. We also have to change limits.

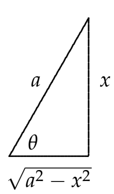
$$\begin{aligned} x = 1 = \sec\theta \quad &\Rightarrow \quad \theta = 0 \\ x = 2 = \sec\theta \quad &\Rightarrow \quad \cos\theta = \tfrac{1}{2} \quad \Rightarrow \quad \theta = \pi/3 \end{aligned}$$

Now proceed with the substitution:

$$\begin{aligned} \int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3} \tan^2\theta\,d\theta &= \int_0^{\pi/3} \sec^2\theta - 1\,d\theta \\ &= \tan\theta - \theta\Big|_0^{\pi/3} \\ &= \sqrt{3} - \pi/3 \end{aligned}$$

Case 3: Integrals involving $\sqrt{a^2 - x^2}$

This was situation in the semi-circle example. In this case, a must correspond to the hypotenuse of the right triangle. (Why?) Again we have our choice of how to label the legs, one side $\sqrt{a^2 - x^2}$ and the other x. With the selection made below, $x = a\sin\theta$. In the circle example we chose to label the legs in the other way because of the geometry involved. Obviously it worked out, but it required us to carry along a minus sign throughout the problem. The choice below is usually simpler.



$$x = a\sin\theta$$
$$dx = a\cos\theta\,d\theta$$
$$\sqrt{a^2 - x^2} = a\cos\theta$$

EXAMPLE 7.3.4. Calculate the following indefinite integral: $\displaystyle\int \frac{x^2}{\sqrt{25-x^2}}\,dx$.

Use the substitution above with $a^2 = 25$ or $a = 5$. We use a reduction formula (see the **Appendix** to these notes) to do the integral.

$$\begin{aligned} \int \frac{x^2}{\sqrt{25-x^2}}\,dx = \int \frac{25\sin^2\theta}{5\cos\theta}\cdot 5\cos\theta\,d\theta = 25\int \sin^2\theta\,d\theta &= 25\int \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\,d\theta \\ &= \tfrac{25}{2}\theta - \tfrac{25}{4}\sin(2\theta) + c \end{aligned}$$

To solve for θ, look back at the original triangle where $\frac{x}{5} = \sin\theta \Rightarrow \arcsin\frac{x}{5} = \theta$ and $\frac{\sqrt{25-x^2}}{5} = \cos\theta$. The simplest way to finish the problem is to make use of a **Double Angle Formula**:

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

Then

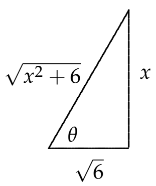
$$\begin{aligned}\tfrac{25}{2}\theta - \tfrac{25}{4}\sin(2\theta) + c = \tfrac{25}{2}\theta - \tfrac{25}{4}(2\sin\theta\cos\theta) + c &= \tfrac{25}{2}\arcsin\tfrac{x}{5} - \tfrac{25}{2}\cdot\tfrac{x}{5}\cdot\tfrac{\sqrt{25-x^2}}{5} + c \\ &= \frac{25}{2}\arcsin\frac{x}{5} - \frac{1}{2}\cdot x\sqrt{25-x^2} + c.\end{aligned}$$

7.4 *Additional Examples*

In this last section we extend the triangle substitution idea to integrals that at first don't appear to have anything to do with 'triangles' because a square root does not immediately appear in them.

EXAMPLE 7.4.1. Calculate the following indefinite integral: $\displaystyle\int \frac{1}{(x^2+6)^{3/2}}\,dx$.

Well there is sort of a square root lurking in the background here. In this case, $(x^2+6)^{3/2}$ may be thought of as $(\sqrt{x^2+6})^3$. So the appropriate triangle is:



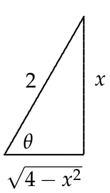
$$x = \sqrt{6}\tan\theta$$
$$dx = \sqrt{6}\sec^2\theta\,d\theta$$
$$\sqrt{x^2+6} = \sqrt{6}\sec\theta$$

Notice that one of the sides of the triangle is $\sqrt{6}$; we may not always have perfect squares. Using the triangle substitution in the box we have:

$$\begin{aligned}\int \frac{1}{(x^2+6)^{3/2}}\,dx = \int \frac{\sqrt{6}\sec^2\theta}{(\sqrt{6}\sec\theta)^3}\,d\theta = \int \frac{\sqrt{6}\sec^2\theta}{6\sqrt{6}\sec^3\theta}\,d\theta = \tfrac{1}{6}\int \frac{1}{\sec\theta}\,d\theta &= \tfrac{1}{6}\int \cos\theta\,d\theta \\ &= \tfrac{1}{6}\sin\theta + c \\ &= \frac{x}{6\sqrt{x^2+6}} + c.\end{aligned}$$

EXAMPLE 7.4.2. Determine the following indefinite integral: $\displaystyle\int \frac{4}{4-x^2}\,dx$.

This time we have to think of $4-x^2$ as $(\sqrt{4-x^2})^2$ to use a triangle.



$$x = 2\sin\theta$$
$$dx = 2\cos\theta\,d\theta$$
$$\sqrt{4-x^2} = 2\cos\theta$$

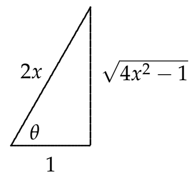
Now we can rewrite the integral.

$$\begin{aligned}\int \frac{4}{4-x^2}\,dx = \int \frac{4}{(2\cos\theta)^2}\cdot 2\cos\theta\,d\theta = \int \frac{8\cos\theta}{4\cos^2\theta}\,d\theta &= 2\int \sec\theta\,d\theta \\ &= 2\int \ln|\sec(\theta)+\tan\theta|\,d\theta \\ &= 2\ln|\tfrac{2}{\sqrt{4-x^2}} + \tfrac{x}{\sqrt{4-x^2}}| + c \\ &= 2\ln|\tfrac{2+x}{\sqrt{4-x^2}}| + c \\ &= \ln|\tfrac{(2+x)^2}{4-x^2}| + c \\ &= \ln|\tfrac{2+x}{2-x}| + c \\ &= \ln|2+x| - \ln|2-x| + c.\end{aligned}$$

We will see this same integral again, shortly, and it will be solved very differently.

EXAMPLE 7.4.3. Evaluate the definite integral: $\displaystyle\int_{1/2}^{1} \frac{x^3}{\sqrt{4x^2-1}}\,dx$.

This time the triangle is obvious, but care is required to label the sides correctly.



$$x = \tfrac{1}{2}\sec\theta$$
$$dx = \tfrac{1}{2}\sec\theta\tan\theta\,d\theta$$
$$\sqrt{4x^2-1} = \tan\theta$$

You could do the indefinite integral and convert back to x to avoid changing the limits. But let's actually change them

$$\begin{aligned}
x &= \tfrac{1}{2} = \tfrac{1}{2}\sec\theta &&\Rightarrow 1 = \sec\theta \Rightarrow && \theta = 0 \\
x &= 1 = \tfrac{1}{2}\sec\theta &&\Rightarrow 2 = \sec\theta \Rightarrow && \cos\theta = \tfrac{1}{2} \quad\Rightarrow\quad \theta = \pi/3
\end{aligned}$$

Now proceed with the substitution (and use the guidelines for integrating powers of $\sec\theta$):

$$\begin{aligned}
\int_{1/2}^{1} \frac{x^3}{\sqrt{4x^2-1}}\,dx = \int_0^{\pi/3} \frac{(\frac{1}{2}\sec\theta)^3}{\tan\theta}\cdot\tfrac{1}{2}\sec\theta\tan\theta\,d\theta &= \frac{1}{16}\int_0^{\pi/3} \sec^4\theta\,d\theta \\
&= \frac{1}{16}\int_0^{\pi/3} \sec^2\theta\cdot\sec^2\theta\,d\theta \\
&= \frac{1}{16}\int_0^{\pi/3} (1+\tan^2\theta)\cdot\sec^2\theta\,d\theta \\
&= \frac{1}{16}\int_0^{\pi/3} \sec^2\theta + \overbrace{\tan^2\theta}^{u^2}\,\overbrace{\sec^2\theta}^{du}\,d\theta \\
&= \frac{1}{16}\left[\tan\theta + \frac{\tan^3\theta}{3}\right]\Bigg|_0^{\pi/3} \\
&= \frac{1}{16}\left[\sqrt{3} + \frac{(\sqrt{3})^3}{3} - 0\right] = \frac{\sqrt{3}}{8}
\end{aligned}$$

7.5 *Problems*

1. Try these problems. A variety of techniques are required, not just triangle substitutions.

(a) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$ (b) $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$

(d) $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ (e) $\displaystyle\int \frac{1}{x\sqrt{x^2-9}}\,dx$ (f) $\displaystyle\int x\sqrt{1-x^2}\,dx$

(g) $\displaystyle\int \frac{4}{4+x^2}\,dx$ (h) $\displaystyle\int \frac{4x}{4+x^2}\,dx$ (i) $\displaystyle\int \frac{4}{4-x^2}\,dx$

(j) $\displaystyle\int \frac{\sqrt{25+x^2}}{x^4}\,dx$ (k) $\displaystyle\int_0^5 \frac{x}{\sqrt{25+x^2}}\,dx$ (l) $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$

(m) $\displaystyle\int \frac{x^3}{\sqrt{25-x^2}}\,dx$ (n) $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,dx$

(o) $\displaystyle\int_{-5}^{0} \sqrt{25-x^2}\,dx$ (p) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

2. Find the arc length of the parabola $f(x) = \frac{x^2}{2}$ on the interval $[0,1]$. You will have to use a trig substitution. Make sure that you switch the limits of the integration.

Answers

1. Caution. . . , it's easy to have made a typo in these answers. Remember: $+c$

(*a*) $\sqrt{x^2-9} - 3\arctan\left(\frac{\sqrt{x^2-9}}{3}\right)$

(*b*) $\dfrac{x}{\sqrt{1+x^2}}$

(*c*) $\arcsin(x/4)$

(*d*) $\frac{\pi}{2} - 1$

(*e*) $\frac{1}{3}\arctan\left(\frac{\sqrt{x^2-9}}{3}\right)$

(*f*) $-\dfrac{(1-x^2)^{3/2}}{3}$

(*g*) $2\arctan(x/2)$

(*h*) $2\ln|4+x^2|$

(*i*) $2\ln\left|\dfrac{2+x}{\sqrt{4-x^2}}\right|$

(*j*) $-\dfrac{1}{75}\left(\dfrac{\sqrt{25+x^2}}{x}\right)^3$

(*k*) $5(\sqrt{2}-1)$

(*l*) $\arcsin(x/5)$

(*m*) $-25\sqrt{25-x^2} + \dfrac{(25-x^2)^{3/2}}{3}$

(*n*) $\ln\left|\dfrac{\sqrt{25+x^2}}{5} + \dfrac{x}{5}\right|$

(*o*) $25\pi/4$

(*p*) $\dfrac{\sqrt{x^2-25}}{25x}$

2. Answer: $\frac{\sqrt{2}+\ln|\sqrt{2}+1|)}{2}$

7.6 *Appendix: Common Trigonometric Formulas and Antiderivatives*

Below are listed several integral formulas for various powers of trig functions.

1. Degree 2 Sine and Cosine Functions. One simple way to do these is to use trig identities. **Know these.**

(*a*) $\displaystyle\int \cos^2 u\, du = \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2u\, du.$

(*b*) $\displaystyle\int \sin^2 u\, du = \int \tfrac{1}{2} - \tfrac{1}{2}\cos 2u\, du.$

2. Low Powers of the Tangent and Secant Functions. These are done with simple identities. **Know these.**

(*a*) $\displaystyle\int \tan u\, du = \int \frac{\sin u}{\cos u}\, du = \ln|\sec u| + c.$

(*b*) $\displaystyle\int \tan^2 u\, du = \int \sec^2 u - 1\, du = \tan u - u + c.$

(*c*) $\displaystyle\int \sec u\, du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\, du = \ln|\sec u + \tan u| + c.$

3. Useful **Double Angle Formula**: $\sin(2\theta) = 2\sin\theta\cos\theta$.

4. Reduction Formulas for Large Powers. These are verified using integration by parts. Repeated application may be necessary.

(*a*) $\displaystyle\int \cos^n u\, du = \tfrac{1}{n}\cos^{n-1} u \sin u + \tfrac{n-1}{n}\int \cos^{n-2} u\, du$

(*b*) $\displaystyle\int \sin^n u\, du = -\tfrac{1}{n}\sin^{n-1} u \cos u + \tfrac{n-1}{n}\int \sin^{n-2} u\, du$

(*c*) $\displaystyle\int \tan^n u\, du = \tfrac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

(*d*) $\displaystyle\int \sec^n u\, du = \tfrac{1}{n-1}\sec^{n-2} u \tan u + \tfrac{n-2}{n-1}\int \sec^{n-2} u\, du$

5. Degree 2 Sine and Cosine Functions Again. If we apply the reduction formulas as for the sine and cosine functions when $n = 2$, we get a different form of the earlier answer. These new forms are better to use with indefinite integrals involving triangle substitutions because it is easier to convert back from u to the original variable x. (Know either these or those in #1.)

(*a*) $\displaystyle\int \cos^2 u\, du = \tfrac{1}{2}\cos u \sin u + \tfrac{1}{2}\int 1\, du = \tfrac{1}{2}\cos u \sin u + \tfrac{1}{2}u + c$

(*b*) $\displaystyle\int \sin^2 u\, du = -\tfrac{1}{2}\sin u \cos u + \tfrac{1}{2}\int 1\, du = -\tfrac{1}{2}\sin u \cos u + \tfrac{1}{2}u + c$