RESULT 3.1. Let $a, b \in \mathbb{Z}$. Then $a^2 + b^2$ is odd if and only if a and b have opposite parity.

(*a*) To get started, fill in the table. This biconditional requires two proofs.

Part 1: "If, then" form:		
Method of Proof	Assumption or first step or hypothesis of the Proof	Goal (What you must show in the proof)
Direct		
Contraposition		
Part 2: "If, then" form:		
Method of Proof	Assumption or first step or hypothesis of the Proof	Goal (What you must show in the proof)
Direct		
Contraposition		

(*b*) Select the methods you think most appropriate and give your proof here.

1	Yifan	Will	Wenshi	
2	Stone	Pat	Matt	
3	Lu	Lauren	Katherine	
4	Adam	Joseph	Hao	
5	Jasmine	Jacque	Doris	
6	Connor	August	Anthony	Josh

RESULT 3.2. (Exercise 3.35.) If n is a non-negative integer, then $2^n + 6^n$ is an even integer.

☞ Hint: There are two cases, but they are not "odd" and "even."

Test Topics

Everything we have covered, through contraposition. Everything through Chapter 3.3 except Chapter 2.11. Review homework and their solutions, and journal problems. Review the examples in the text

1. Basic set language definitions: subset, proper subset, intersection, union, difference, complement, index sets (unions, intersections), partitions, Cartesian products, power sets.
2. Logic: statements, open sentences; negation of statements including with quantifiers; disjunctions, conjunctions, implications, biconditionals (all with negations); logical equivalence; using truth tables; fundamental properties of logical equivalence—the "algebra of logical relations" (DeMorgan's Laws, distributivity, etc,); translating symbolic statements into English and vice versa; putting statements into "If, then" form; determining for which elements in the domain are quantified statements true; other ways to write $P \Rightarrow Q$; the difference between the converse, the contrapositive, and the negation of an implication; expressing implications and their negations using disjunctions and conjunctions.
3. Proof Techniques: Vacuous, Trivial, Direct, Direct Proof via Lemma, Contraposition. (Definitions of odd and even. You may use basic properties of reals (squares are non-negative; basic facts about trig, log, and exponential functions of reals), but be sure to justify other claims.

Proof by Cases

Think about what you did in the homework proof in which you examined what happened for each element in the set $S = \{0, 1, 2\}$. You dealt with three separate cases and showed that the result held in each (two of them vaucuously). We formalize this process.

Proof by Cases:

- Partition (what does that mean?) the domain S of the hypothesis into subsets each of which has a common property.
- For each subset, use its defining property to (help) verify the result for that particular subset.
- Completing this for all subsets in the partition of S proves the result for all of S.

Note: We may even further divide a particular case into **subcases**. There are a few natural situations where this might arise.

- To prove: For all $n \in \mathbb{Z}, P(n) \Rightarrow Q(n)$. It might be useful to divide the proof into two cases:

  Case 1. Assume n is odd.

  Case 2. Assume n is even.

- Or suppose that you are trying to prove: For all $x \in \mathbb{R}, P(x) \Rightarrow Q(x)$. It might be useful to divide the proof into three cases:

  Case 1. Assume $x < 0$.

  Case 2. Assume $x = 0$.

  Case 3. Assume $x > 0$.

Examples

DEFINITION 3.4.1. Two integers a and b have the **same parity** if a and b are both even or are both odd. Similarly, a and b have **opposite parity** if one of them is even and the other is odd.

RESULT 3.3. If $n \in \mathbb{Z}$, then n and n^2 have the same parity.

Proof. Assume $n \in \mathbb{Z}$. (Show n and n^2 have the same parity.) There are two cases to consider: (1) n is odd; (2) n is even.

Why do cases make sense here? Is the contrapositive helpful?

Case 1. Assume n is odd. Then $n = 2a + 1$ for some $a \in \mathbb{Z}$. So

$$n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1.$$

Since $2a^2 + 2a$ is an integer, n^2 is odd and has the same parity as n.

Case 2. Assume n is even. Then $n = 2b$ for some $b \in \mathbb{Z}$. So

$$n^2 = 4b^2 = 2(2b).$$

Since $2b$ is an integer, n^2 is even and has the same parity as n.

□

ANALYSIS: There are several other ways to state this same result. Why is it equivalent to the following biconditionals?

Once you know the parity of one of the terms, you know the parity of the other.

1. Let $n \in \mathbb{Z}$. Then n is even if and only if n^2 is even.
2. Let $n \in \mathbb{Z}$. Then n is odd if and only if n^2 is odd. (Contrapositive of (1))

Result 3.3 says once you know the parity of one of the quantities you automatically know the parity of the other. ♢

RESULT 3.4. Let $a, b \in \mathbb{Z}$. The integers a and b have the same parity if and only if $a - b$ is even.

Proof. This is a biconditional so there are two results to prove.

$\Rightarrow$: First assume that a and b have the same parity. (Show that $a - b$ is even.) There are two cases to consider.

Case 1. Assume a, b are odd. Then $a = 2x + 1$ and $b = 2y + 1$ for some $x, y \in \mathbb{Z}$. So

$$a - b = (2x + 1) - (2y + 1) = 2(x - y).$$

Since $x - y$ is an integer, $a - b$ is even.

☡ Why can't we just say $a = 2x + 1$ and $b = 2x + 1$ for some $x \in \mathbb{Z}$?

Case 2. Assume a, b are even. Then $a = 2x$ and $b = 2y$ for some $x, y \in \mathbb{Z}$. So

$$a - b = 2x - 2y = 2(x - y).$$

Since $x - y$ is an integer, $a - b$ is even.

$\Leftarrow$: (Use contraposition to prove the converse.) Assume a and b have opposite parity. (Show $a + b$ is odd.) There are two cases two consider:

Case 1. Assume a is even and b is odd. Then $a = 2x + 1$ and $b = 2y$ for some $x, y \in \mathbb{Z}$. So

$$a - b = (2x + 1) - 2y = 2(x - y) + 1.$$

Since $2(x - y)$ is an integer, $a - b$ is odd.

Case 2. Assume a is odd and b is even. The proof is similar to the first case.

Since if a and b have the same parity, then $a - b$ is even and conversely, then the biconditional is true.

□

ANALYSIS: When a part of a proof repeats an earlier argument, note that. Do not rewrite it. Here's another proof of the converse:

Proof. $\Leftarrow$: To prove the converse we use contraposition. Assume that a and b have opposite parity, one is even and the other is odd. Without loss of generality (WLOG) we may assume a is even and b is odd. Then $a = 2x + 1$ and $b = 2y$ for some $x, y \in \mathbb{Z}$. So

$$a - b = (2x + 1) - 2y = 2(x - y) + 1.$$

Since $x - y$ is an integer, $a - b$ is odd. □

This makes sense: a and b are both integers where one is even and the other odd. That's the only property we use in the proof. ♢

RESULT 3.5. Let $a, b \in \mathbb{Z}$. Then $a^2 + b^2$ is odd if and only if a and b have opposite parity.

Proof. This is a biconditional so there are two results to prove.

$\Rightarrow$: First assume that a and b have opposite parity. (Show that $a^2 + b^2$ is odd.) WLOG we may assume a is even and b is odd. Then Then $a = 2x + 1$ and $b = 2y$ for some $x, y \in \mathbb{Z}$. So

$$a^2 + b^2 = (2x + 1)^2 - (2y)^2 = 4x^2 + 4x1 + 4y^2 = 2(2x^2 + 2x + 2y^2) + 1.$$

Since $2x^2 + 2x + 2y^2$ is an integer, $a^2 + b^2$ is odd. □

$\Leftarrow$: (Contraposition.) Assume a and b have the same parity. (Show $a^2 + b^2$ is even.) There are two cases to consider.

Case 1. Assume a, b are odd. Then $a = 2x + 1$ and $b = 2y + 1$ for some $x, y \in \mathbb{Z}$. So

$$a^2 + b^2 = (2x+1)^2 + (2y+1)^2 = 4x^2 + 4x + 1 + 4y^2 + 4y + 1 = 2(2x^2 + 2x + 2y^2 + 2y + 1).$$

Since $2x^2 + 2x + 2y^2 + 2y + 1$ is an integer, $a^2 + b^2$ is even.

Case 2. Assume a, b are even. Then $a = 2x$ and $b = 2y$ for some $x, y \in \mathbb{Z}$. So

$$a^2 + b^2 = 4x^2 + 4y^2 = 2(2x^2 + 2y^2).$$

Since $2x^2 + 2y^2$ is an integer, $a^2 + b^2$ is even.

DEFINITION 3.4.2. Let $x \in \mathbb{R}$. The the **absolute value of** x is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

RESULT 3.6. Let $a, b \in \mathbb{R}$, with $b \geq 0$. If $|a| \leq b$, then $-b \leq a \leq b$.

Proof. Let $a, b \in \mathbb{R}$, with $b \geq 0$. Assume $|a| \leq b$. (Show $-b \leq a \leq b$.) Since $b \geq 0$, it follows that $-b \leq 0$. Since absolute values are non-negative, it follows that $-b \leq |a|$. Since $|a| \leq b$, we now have

$$-b \leq |a| \leq b. \tag{3.1}$$

There are two cases to consider.

Case 1. Assume $a \geq 0$. By definition of absolute value, $|a| = a$. Substituting into (3.1) gives $-b \leq a \leq b$. So the implication holds in this case.

Case 2. Assume $a < 0$. By definition of absolute value, $|a| = -a$. Substituting into (3.1) gives $-b \leq -a \leq b$. Multiplying by -1 changes the direction of the inequalities, so $b \geq a \geq -b$. So the implication also holds in this case.

□