Working with Logical Equivalences

Some of the less obvious key logical equivalences:

1. $[P \Rightarrow Q] \equiv [(\sim P) \vee Q]$. We can rewrite an implication as a disjunction (and a disjunction as an implication).

2. Distributivity

(*a*) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

(*b*) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R$.

3. DeMorgan's laws

(*a*) $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$.

(*b*) $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

Other ways to say $P \Rightarrow Q$. An implication can take many different forms:

(1) If P, then Q. (Preferred: makes hypothesis and conclusion clear.)

(2) Q if P.

(3) P implies Q. (Also clear.)

(4) P only if Q. (Less clear. When we have P, then we get Q.)

(5) P is sufficient for Q. (For Q, it is sufficient that P.) Yes, English is confusing.

(6) Q is necessary for P. (For P, it is necessary that Q.)

4. Let P: It is raining. Let Q: It is cloudy.

(*a*) State (in words): $P \Rightarrow Q$.

(*b*) State the implication using "necessary."

(*c*) State the implication using "only if."

(*d*) State as a logically equivalent disjunction: $P \Rightarrow Q$.

(*e*) State $\sim [P \Rightarrow Q]$ as a conjunction. (What properties are you using?)

(*f*) State $\sim [P \wedge (\sim Q)]$. Then convert to an implication.

(*g*) State the converse to the originally implication.

5. Let P: It is snowing. Let Q: It is cold. Same questions as above.

6. Let P: It is snowing. Let Q: The game is cancelled.

(*a*) State (in words): $P \iff Q$.

(*b*) State $P \iff Q$ as a conjunction of implications. (Then state in words.)

(*c*) State the negation of $P \iff Q$ using only $\sim$, $\wedge$, and $\vee$. And then state in words

(*d*) What biconditional is the negation of: "It is snowing and the game is outside or it is not snowing and the game is inside."

EXERCISE 2.10.1. Try these.

(*a*) Let $n \in \mathbb{Z}$. Negate: "If $5n + 2$ is even, then $3n + 1$ is odd." using a conjunction.

(*b*) Let $n \in \mathbb{Z}$. Write the negation of: "n^3 is even and n odd." as an implication.

(*c*) Let $n \in \mathbb{Z}$. "$n + 7$ is odd and n^2 is even." is the negation of which implication?

(*d*) Let $n \in \mathbb{Z}$. What biconditional is the negation of: "$5n^2$ is odd and $n^3 + 1$ is even or "$5n^2$ is even and $n^3 + 1$ is odd "

(*e*) Let $n \in \mathbb{Z}$. What biconditional is the negation of: "$n^2 + 3$ and $3n + 9$ are odd or $n^2 + 3$ and $3n + 9$ are even."

4. Let P: It is raining. Let Q: It is cloudy.

(*a*) State (in words): $P \Rightarrow Q$. **Solution:** If it is raining, then it is cloudy.

(*b*) State the implication using "necessary." **Solution:** To be rainy, it is necessary to be cloudy. It is necessary to be cloudy for it to rain.

(*c*) State the implication using "only if." **Solution:** It is raining only if it is cloudy.

(*d*) State as a logically equivalent disjunction: $P \Rightarrow Q$. **Solution:** It is not raining or it is cloudy. $(\sim P) \vee Q$.

(*e*) State $\sim [P \Rightarrow Q]$ as a conjunction. (What properties are you using?) **Solution:** It is raining and it is not cloudy. $P \wedge \sim Q$.

(*f*) State $\sim [(\sim P) \wedge Q]$. Then convert to an implication. **Solution:** $P \vee (\sim Q)$. It is raining or it is not cloudy. If it is not raining, then it is not cloudy. $(\sim P) \Rightarrow [\sim (\sim Q)]$.

(*g*) State the converse to the originally implication. **Solution:** If it is cloudy, then it is raining.

5. Let P: It is snowing. Let Q: It is cold. Same questions as above.

6. Let P: It is snowing. Let Q: The game is cancelled.

(*a*) State (in words): $P \iff Q$. **Solution:** It is snowing if and only if the game is cancelled.

(*b*) State $P \iff Q$ as a conjunction of implications. (Then state in words.) **Solution:** $(P \Rightarrow Q) \wedge Q \Rightarrow P$. If it is snowing, then the game is cancelled and if the game is cancelled, then it is snowing.

(*c*) State the negation of $P \iff Q$ in words without using implication. **Solution:** It is not snowing AND the game is cancelled or the game is played AND it is snowing. Better: It is not snowing AND the game is cancelled or it is snowing AND the game is played.

(*d*) What biconditional is the negation of: "It is snowing and the game is outside or it is not snowing and the game is inside." **Solution:** $[(\sim P) \wedge Q] \vee [P \wedge (\sim Q)]$. So we could let P: It is not snowing and $\sim Q$: The game is outside. Its negation is the biconditional $P \iff Q$. It is not snowing if and only if the game is inside.