Assignment 10. Problems 1 to 5 due Monday. The others are due Wednesday. I will add a couple more on Monday.

1. Section 2.2, Exercises 2, 4, and 7. For 7(a) see Example 4 in the text. For 7(b) reduce the entire "super augmented" matrix $[A\,\mathbf{b}_1\,\mathbf{b}_2\,\mathbf{b}_3\,\mathbf{b}_4]$ all at once. Check that the solutions you found in part (a) appear in the last four columns.

2. Suppose that A is an $n \times n$ invertible matrix and c is a non-zero scalar. Show that cA is invertible by finding a formula (similar to those of Theorem 2.6) for $D = (cA)^{-1}$. Verify that your formula works; check that $D(cA) = I$ and $(cA)D = I$.

3. Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with standard matrix A. Prove: If A is invertible, then T is a one-to-one. Hint: One method is to use the One-to-One Dictionary to show that if A is invertible, then A satisfies one of the other conditions of equivalent to T being one-to-one. (Look at the theorems we proved Friday.)

Instructions. These next three problems use basic properties of matrix multiplication. You need to be aware that you are using such properties, because it is easy to make mistakes. Matrix multiplication is more complicated than ordinary scalar multiplication. With this in mind: Justify matrix any multiplication properties used by referring to an appropriate theorem in Section 2.1 or 2.2. E.g., "$A(BC) = (AB)C$ by associativity (Theorem 2.1(a))" or "$IC = C$ by Theorem 2.1 (e)" or "Since A is invertible, $(A^{-1})^{-1} = A$ by Theorem 2.6. Each time you multiply mention the side. E.g., "Left-multiply both sides of the equation by C." If you multiply by the inverse of a matrix, mention why you know the inverse exists. E.g., "Since D is the product of invertible matrices, by Theorem 2.6, D is invertible. So we can right-multiply by D^{-1}." Or "We are given that C is invertible, so C^{-1} exists." Here's an example:

EXAMPLE 0.0.1. Suppose that $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is an $n \times n$ invertible matrix. Prove that $B = C$.

Proof. Given $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible matrix. (Show $B = C$.) Since D is invertible, D^{-1} exists so we can right-multiply by D^{-1} to get:

$$(B - C)(DD^{-1}) = 0D^{-1} = 0.$$

By definition of inverse, $DD^{-1} = I$ so we get $(B - C)I = 0$. Since I is the identity matrix, by Theorem 2.2 (e),

$$(B - C)I \overset{2.2(e)}{=} B - C = 0, \text{ so } B = C + 0 = C \text{ by Theorem 2.1 (c).}$$

□

4. Section 2.2, Exercise 8.

5. Section 2.2, Exercise 12. This proof should be short, but make sure you justify each step very precisely.

——— Wednesday ———

6. Section 2.2, Exercise 16. Note that you CANNOT say, "since AB is invertible, then $(AB)^{-1} = B^{-1}A^{-1}$," because you do not know that A^{-1} exists — you are trying to PROVE that A is invertible. (You can only apply the Socks-Shoes Theorem when you already know that the two matrices in question are both invertible.) After you use the Hint in the text, you will find that A equals some other matrix. Explain why this other matrix is invertible. So what can you conclude about A?

7. Find the inverse of the matrix D in Exercise 11 of Section 2.1 and the inverse of the matrix in Section 2.2, Exercise 32. Show all work.

8. ☞ These problems concern the determinant of 2×2 matrices. Use the formula after Theorem 4 in the text (p 103).

(*a*) Prove the following. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary 2×2 matrix and let r be a scalar. Then $\det(rA) = r^2 \det A$. Just write rA and apply the formula.

(*b*) Prove the following. Let A be an arbitrary 2×2 matrix. Then $\det A^T = \det A$.

(*c*) Bonus: Prove the following. Let A be an arbitrary 2×2 **invertible** matrix. Then $\det A^{-1} = \dfrac{1}{\det A}$.

9. (*a*) Which of the following matrices are elementary matrices?

$$(1)\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad (4)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (5)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(*b*) For each of the elementary matrices above, determine its inverse using the idea at the top of page 107.

(*c*) The matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the product of two elementary matrices E_2E_1.

Determine those two matrices and then use them to determine A^{-1}. Careful: Remember "Socks and shoes."