Assignment 11

A few straightforward problems on determinants will be added on Friday.

1. (Review problem.) Consider the matrix $C = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Assume that the matrix C is the standard matrix for transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(*a*) What are the values of n and m?

(*b*) Is T one-to-one? Justify your answer very clearly, citing specific theorems.

(*c*) Is Tonto $\mathbb{R}^m$? Justify your answer very clearly, citing specific theorems.

2. Prove each of the following statements. Be sure you cite theorems to justify your claims. If you use the Connections Theorem, state explicitly which parts you are using as follows. Example: "Since A^T is invertible, by the Connections Theorem A the columns of A are independent."

Harsh grading if there is no justification!

(*a*) If A is an $n \times n$ matrix and $\mathbf{c} \in \mathbb{R}^n$ is a vector so that $A\mathbf{x} = \mathbf{c}$ is INconsistent, then A's columns are linearly dependent.

(*b*) If A is an $n \times n$ matrix and $\mathbf{c} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{c}$ has more than one solution, then A's columns do not span $\mathbb{R}^n$.

(*c*) If A and B are $n \times n$ matrices such that each of $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $AB\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

(*d*) If A can be written as a product of elementary matrices, then A's columns are linearly independent.

(*e*) If $n \times n$ matrix A is invertible, then the ROWS of A span $\mathbb{R}^n$.

3. One of these things is not like the others: In three of the following, it is impossible to give an example that meets the stated criteria, while in the fourth, an example is possible. Give an example in the one case where it is possible to do so, and PROVE that your example fits the bill. For the remaining three cases, PROVE that it is impossible to give an example. Your proofs should be short but rigorous. Make sure you cite theorems to justify your claims; if you use the Connections Theorem, state explicitly which parts you are using in the style described in the previous problem.

(*a*) A linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ that is one-to-one but not onto $\mathbb{R}^5$.

(*b*) A 5×5 non-singular matrix A whose transpose is the product of elementary matrices.

(*c*) A 5×5 matrix A with 5 pivots such that A's first column is a linear combination of A's second and third columns.

(*d*) A onto linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ whose standard matrix has an all-zero column.

4. (This is not the same problem as on part 2 of Assignment 10.) Let A and B be $n \times n$ matrices. Prove: If AB is invertible, then so is A. (Note: You cannot use the Socks-Shoes Theorem and write $(AB)^{-1} = B^{-1}A^{-1}$, because the Socks-Shoes Theorem only applies when BOTH A and B are known to be invertible.) Big Hint: Here's one method. Use the definition of invertible from class on Wednesday February 24 (or see page 103) to obtain a useful matrix C. Then multiply AB by C (which side?) and use associativity and then the Connections Theorem to get the result.

5. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & \frac{1}{2} & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

6. Section 2.3, page 116, Exercise 36.

7. Section 3.1, page 167, Exercise 4. Read the instructions. Do it two ways.

8. Section 3.1, page 168, Exercise 20, 22, 24. Read the instructions. This is an important problem.

9. **Extra Credit.** Assume A is an $n \times n$ invertible matrix that is the standard matrix for a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Prove: If $\mathbf{v}$ is in the kernel of T, then $\mathbf{v} = \mathbf{0}$.