Assignment 12

Due Wednesday.

Name		
Problem	**Points**	**Score**
1	20	
2	12	
Total	32	

1. This question reviews several ideas. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Assume that $T(\mathbf{u}) = T\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $T(\mathbf{v}) = T\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(*a*) Determine $T(3\mathbf{u} - 2\mathbf{v})$. Avoid using matrix multiplication.

SOLUTION. $T(3\mathbf{u} - 2\mathbf{v}) \overset{\text{linear}}{=} 3T(\mathbf{u}) - 2T(\mathbf{v}) = 3\begin{bmatrix} 6 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ -8 \end{bmatrix}$.

(*b*) Determine $T(\mathbf{e}_1)$. Hint: Express $\mathbf{e}_1$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Then proceed as in part (a).

SOLUTION. We first need to solve

$$\begin{matrix} 2x_1 + 4x_2 = 1 \\ 4x_1 + 6x_2 = 0 \end{matrix} \text{ or } \begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 1 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -3 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

So $T(\mathbf{e}_1) = T(-\frac{3}{2}\mathbf{u} + \mathbf{v}) = -\frac{3}{2}\begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

(*c*) Similarly determine $T(\mathbf{e}_2)$.

SOLUTION. This time

$$\begin{matrix} 2x_1 + 4x_2 = 0 \\ 4x_1 + 6x_2 = 1 \end{matrix} \text{ or } \begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

So $T(\mathbf{e}_2) = T(\mathbf{u} - \frac{1}{2}\mathbf{v}) = \begin{bmatrix} 6 \\ -2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{bmatrix}$

(*d*) Determine the standard matrix A for T.

SOLUTION. $A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} -8 & \frac{11}{2} \\ 4 & -\frac{5}{2} \end{bmatrix}$.

(*e*) Is A invertible?

SOLUTION. Yes, since $\det A = -2 \neq 0$ So $A^{-1} = -\frac{1}{2}\begin{bmatrix} -\frac{5}{2} & -\frac{11}{2} \\ -4 & -8 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{11}{4} \\ 2 & 4 \end{bmatrix}$.

(*f*) Read about invertible transformations on page 113–114 and give the standard matrix for the inverse transformation T^{-1}.

SOLUTION. Because A is invertible T is invertible and its standard matrix is So $A^{-1} = -\frac{1}{2}\begin{bmatrix} -\frac{5}{2} & -\frac{11}{2} \\ -4 & -8 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{11}{4} \\ 2 & 4 \end{bmatrix}$.

(*g*) Determine all the vectors $\mathbf{w}$ such that $T(\mathbf{w}) = \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

SOLUTION. We need to solve $T(\mathbf{w}) = \mathbf{b}$ or equivalently $A\mathbf{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So

$$\mathbf{w} = A^{-1}A\mathbf{w} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{5}{4} & \frac{11}{4} \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 4 \end{bmatrix}.$$

(*h*) Determine $T(\mathbf{x}) = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

SOLUTION. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 & \frac{11}{2} \\ 4 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8x_1 + \frac{11}{2}x_2 \\ 4x_1 - \frac{5}{2}x_2 \end{bmatrix}$.

(*i*) Is T onto? Explain carefully citing appropriate theorems.

SOLUTION. Since the $n \times n$ matrix A is invertible, condition (a) of the Connections Theorem is true, so they all are. Therefore the transformation T is onto (condition (h)).

(*j*) Is T one-to-one? Explain carefully citing appropriate theorems.

SOLUTION. Since the $n \times n$ matrix A is invertible, condition (a) of the Connections Theorem is true, so they all are. Therefore the transformation T is one-to-one (condition (l)).

2. (*a*) Section 3.2 page 175, #14. Also decide whether the matrix is invertible.

SOLUTION. First use row 4 to eliminate entries, then row 2. The determinant is zero, so not invertible.

$$\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -6 & 1 & 0 \\ 1 & 3 & 0 & -3 \\ 0 & 0 & -2 & 12 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -6 & 1 & 0 \\ 1 & 3 & 0 & -3 \\ 0 & 0 & -2 & 12 \\ 0 & -13 & 0 & 13 \end{vmatrix} = -1\begin{vmatrix} -6 & 1 & 0 \\ 0 & -2 & 12 \\ -13 & 0 & 13 \end{vmatrix} = -1\begin{vmatrix} -6 & 1 & 0 \\ -12 & 0 & 12 \\ -13 & 0 & 13 \end{vmatrix} = -1(-1)\begin{vmatrix} -12 & 12 \\ -13 & 13 \end{vmatrix} = 0$$

(*b*) Section 3.2 page 175, #18, 20. Read the instructions. Use a theorem, not a calculation to determine the answers.

SOLUTION. EXERCISE 18: There have been two row interchanges so by Theorem 3.3 the determinant is multiplied by $(-1)^2 = 1$. The determinant is unchanged: $\det A = 7$, the original determinant.

EXERCISE 20: Row 2 has been added to row 1, so by Theorem 3.3 the determinant is unchanged: $\det A = 7$, the original determinant.

(*c*) Section 3.2 page 175, #22.

SOLUTION. Expanding along the first row,

$$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 5\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} - 0 - 1\begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix} = 5 - 5 = 0.$$

So the matrix is not invertible by Theorem 3.4.

(*d*) Section 3.2 page 175, #26.

SOLUTION. Since there are 4 vectors in $\mathbb{R}^4$, by the Connections Theorem the vectors will be independent if the determinant of the corresponding 4×4 matrix is not zero. Expanding along the fourth column,

$$\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = -3\begin{vmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{vmatrix}$$
$$= -3\left(-6\begin{vmatrix} 2 & -2 \\ -6 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 5 & -6 \end{vmatrix}\right) = -3(-6(-14) + 3(-28)) = 0.$$

So the columns are not independent.