Reading and Practice

Reread Section 4.1 and Read Section 4.2. The Null Space and Column Space are fundamental vector spaces associated with matrices and their corresponding transformations.

1. Practice Problems. Page 195ff [Check answers in the back.] #1, 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29.

Today's Key Concepts

THEOREM 4.1.1 (Basic Properties of Vector Spaces). Let $\mathbb{V}$ be a vector space and $\mathbf{u} \in \mathbb{V}$.

(*1*) The zero vector (additive identity) $\mathbf{0} \in \mathbb{V}$ is unique. That is, if $\mathbf{u} + \mathbf{w} = \mathbf{u}$ for all $u \in \mathbb{V}$, then $\mathbf{w} = \mathbf{0}$.

(*2*) $0\mathbf{u} = \mathbf{0}$ for any $u \in \mathbb{V}$.

(*3*) $c\mathbf{0} = \mathbf{0}$.

(*4*) The $-\mathbf{u}$ is the unique vector in $\mathbb{V}$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. That is, if $\mathbf{u} + \mathbf{w} = \mathbf{0}$, then $\mathbf{w} = -\mathbf{u}$.

(*5*) For any $u \in \mathbb{V}$, $(-1)\mathbf{u} = -\mathbf{u}$ (i.e., $(-1)\mathbf{u}$ is the additive inverse of $\mathbf{u}$.

DEFINITION 4.1.2. A **subspace** of a vector space V is a subset H of $\mathbb{V}$ that has the following three properties.

(*a*) The zero vector of $\mathbb{V}$ is in H.

(*b*) H is closed under addition: That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

(*c*) H is closed under scalar multiplication: That is, for each $\mathbf{u} \in H$, and each scalar c, the vector $c\mathbf{u} \in H$.

THEOREM 4.1.3 (Subspaces are Vector Spaces). Let H be a subspace of a vector space $\mathbb{V}$. Then H is, itself, a vector space.

You should recognize and be comfortable with the following vector spaces: $\mathbb{R}^n$, $M_{m\times n}$, $\mathbb{P}_n$, $\mathcal{F}$, $\mathcal{F}_\mathcal{D}$ (the functions defined on a domain D).

Hand In Monday

1. (Review problem—like Test 2). Suppose that A and E are $n \times n$ matrices and E is an elementary. Prove: If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A^T \sim E$.

2. EXERCISE 4.1 #2: Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$.

   (-) Begin by giving an example of a vector $\mathbf{w} \in W$.

   (*a*) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u} \in W$? Why?

   (*b*) Find $\mathbf{u}, \mathbf{v} \in W$ such that $\mathbf{u} + \mathbf{v} \notin W$.

3. For these three problems use the definition of a subspace on page 193.

   (*a*) Page 196, Exercise #6: Let $H = \{\mathbf{p} \in \mathbb{P}_2 : \mathbf{p}(t) = a + t^2\}$. (1) Give an explicit example of a vector $\mathbf{q} \in H$. (2) Determine whether H a subspace of $\mathbb{P}_2$.

   (*b*) Page 196, Exercise #8: Let $H = \{\mathbf{p} \in \mathbb{P}_n : \mathbf{p}(0) = 0\}$. (1) Give an explicit example of a non-zero vector $\mathbf{q} \in H$. (2) Determine whether H a subspace of $\mathbb{P}_n$.

   (*c*) Exercise #22, Page 196. Let F be a fixed 3×2 matrix. Determine whether $H = \{A \in M_{2\times 4} : FA = \mathbf{0}_{3\times 4}\}$ a subspace of $M_{2\times 4}$.

For part (a) the condition means that the coefficient of t^2 is 1.

4. Let $\mathbb{K}$ be the set of 2×2 **singular** matrices, i.e., $\mathbb{K} = \{A \in M_{2\times 2} : \det A = 0\}$. Show that $\mathbb{K}$ is NOT a subspace of $M_{2\times 2}$ by giving specific 2×2 matrices to show that one of the subspaces properties fails.

For Wednesday

5. (*a*) Exercise #12, Page 196. Let $W = \left\{ \begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix} \in \mathbb{R}^4 : s, t \text{ scalars} \right\}$. (1) Give an explicit example of a non-zero vector $\mathbf{v} \in W$. (2) Show that W is a subspace of $\mathbb{R}^4$.

(*b*) Exercise #14, Page 196. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is $\mathbf{w}$ is the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(*c*) Exercise #16, Page 196. $W = \left\{ \begin{bmatrix} 1 \\ 3a-5b \\ 3b+2a \end{bmatrix} \in \mathbb{R}^3 : a, b \text{ scalars} \right\}$. Is W is a subspace of $\mathbb{R}^3$?

(*d*) Exercise #18, Page 196. $W = \left\{ \begin{bmatrix} 4a+3b \\ 0 \\ a+3b+c \\ 3b-2c \end{bmatrix} \in \mathbb{R}^4 : a, b, c \text{ scalars} \right\}$. Is W is a subspace of $\mathbb{R}^4$?

6. Let $\mathbb{S}$ be the set of $n \times n$ **symmetric** matrices, i.e., $\mathbb{S} = \{A \in M_{n\times n} : A = A^T\}$. Determine whether $\mathbb{S}$ is a subspace of $M_{n\times n}$.

Thinking ahead. Now that we have defined general vector spaces we can generalize the idea of a linear transformation.

DEFINITION 4.1.4. If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, then $T : \mathbb{V} \to \mathbb{W}$ is a **linear transformation** if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{V}$ and all scalars c.

DEFINITION 4.1.5. $T : \mathbb{V} \to \mathbb{W}$ is **onto** if for every $\mathbf{w} \in \mathbb{W}$, there is at least one vector $\mathbf{v} \in \mathbb{V}$ so that $T(\mathbf{v}) = \mathbf{w}$.

DEFINITION 4.1.6. $T : \mathbb{V} \to \mathbb{W}$ is **one-to-one** if whenever $T(\mathbf{u}) = T(\mathbf{v})$ then $\mathbf{u} = \mathbf{v}$.

7. Let $T : \mathbb{R}^2 \to M_{2\times 2}$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & y \\ y & 2x \end{bmatrix}$

(*a*) Prove T is a linear transformation.

(*b*) Determine if T is one-to-one. (If $T(\mathbf{u}) = T(\mathbf{v})$ must $\mathbf{u} = \mathbf{v}$?)

(*c*) **Bonus**: Determine if T is onto.