Assignment 4 (Answers)

Name		
Problem	**Points**	**Score**
1	10	
2	10	
3	4	
4	15	
5	15	
6	4	
7	5	
8	7	
9	8	
Total	78	

General Comments: Make sure to fully answer the question. For example, the last problem asks you to state what the linear combination actually is while many of you just determined whether the vector was in Span $\{H\}$. Make sure to copy the problem correctly! Most of all, make sure to justify your answers or give a proof when asked.

1. Section 1.2, Exercises 17 and 18. Be sure to show your work and justify your answers. Review the answers to a similar question on the Day 2 Assignment.

Solution. We can tell whether a system is consistent or not by the position of the pivots. So reduce the augmented matrix to EF or RREF and determine whether there is a pivot in the final column.

(*a*) #17. Using $R_2 + 2R_1$ we get

$$\begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & 4 \\ 0 & \boxed{1} & h+8 \end{bmatrix} = EF$$

Since the EF never has a pivot in the final column, by Theorem 2 the system is consistent for any value of h.

(*b*) #18. Again reduce to EF. Clear out below the leading 1 in the first row using $R_2 - hR_1$.

$$\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -3 & 1 \\ 0 & 6+3h & -2-h \end{bmatrix} = EF$$

There are two possibilities: (1) If $6 + 3h \neq 0$, then it is a pivot, so the EF does not have a pivot in the final column and the system is consistent by Theorem 2. (2) If $6 + 3h = 0$, then $h = -2$, so the EF becomes

$$\begin{bmatrix} \boxed{1} & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} = EF.$$

Again there is no pivot in the final column. So the system is consistent by Theorem 2. Thus, the system is consistent for *all* values of h.

2. Section 1.2, Exercise 20. Be sure to show your work/reasoning.

Solution. The EF of the augmented matrix is

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix} = EF$$

(*a*) By Theorem 2 the system has no solution, if there is a pivot in the right most column. This means the second row of the EF must be $\begin{bmatrix} 0 & 0 & k-2 \end{bmatrix}$, with $k - 2 \neq 0$. So from the EF, we must have $h + 6 = 0$, so $h = -6$ and $k \neq 2$.

(*b*) By Theorem 2 the system has a unique solution, if there are no free variables, so $h + 6$ must be a pivot. This means that $h + 6 \neq 0$ or $h \neq -6$. In this case, there is no pivot in the third column, so the system is consistent. Thus, $h \neq -6$ and k is any real yield unique solutions.

(*c*) By Theorem 2 the system has a infinitely many solutions, if there is a free variable and no pivot in the final column. Since x_1 is basic, x_2 must be free, so $h + 6 = 0$ or $h = -6$. To avoid a pivot in the final column means $k - 2 = 0$, so $k = 2$. The EF is now $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The system is consistent, x_2 is free, and so there are infinitely many solutions.

3. Section 1.2, Exercise 24. Be sure to justify your answer.

Solution. The augmented matrix is 3×5 and the fifth column is not a pivot column. The fifth column is the rightmost column. By Theorem 2, since it does not contain a pivot, the system is consistent.

4. For each of the following, decide whether or not it is possible for a system to satisfy the given description. If it is possible, give an augmented matrix (in row-echelon or reduced row-echelon form) that corresponds to such a system and prove that the corresponding system does in fact fulfill the requirements; if it is not possible, prove that it is not possible.

(*a*) A system of 5 equations in 3 unknowns that has exactly 1 solution.

Proof. Possible. Consider the system a system of 5 equations in 3 unknowns with RREF

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Giving an example is not enough. You are asked to **prove** that the example you give works. Use Theorem 2 to justify it.

There is no pivot in the final column, there are pivots in the other columns, so by Theorem 2, the solution is unique. Here the solution is $(0, 0, 0)$ □

(*b*) A system of 5 equations in 3 unknowns that has infinitely many solutions.

Proof. Possible. Consider the system a system of 5 equations in 3 unknowns with RREF

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Giving an example is not enough. You are asked to **prove** that the example you give works. Use Theorem 2 to justify it.

There is no pivot in the final column, there are pivots only in the first two columns, so x_3 is free. By Theorem 2, There are infinitely many solutions. (In this case, they are $x_1 = 0$, $x_2 = 0$, and $x_3 =$ free.) □

(*c*) A system of 5 equations in 3 unknowns that has exactly 2 solutions.

Proof. Impossible. By Theorem 2, there are either no solutions, exactly one solution, or infinitely many solutions. □

5. Repeat Problem 4 for the following statements.

(*a*) A system of 3 equations in 5 unknowns that has infinitely many solutions.

Proof. Possible. Consider the system a system of 3 equations in 5 unknowns with RREF

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Be sure to give your proof. Use Theorem 2 to justify it.

There is no pivot in the final column, there are pivots only in the first three columns, so x_4 and x_5 aare free. By Theorem 2, There are infinitely many solutions. (In this case, they are $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, $x_4 =$ free, $x_5 =$ free.) □

(*b*) A system of 3 equations in 5 unknowns that has no solutions.

Proof. Possible. Consider the system a system of 3 equations in 5 unknowns with RREF

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is a pivot in the final column, so by Theorem 2, the system is not consistent (no solutions). □

Again, be sure to give your proof. Use Theorem 2 to justify it.

(*c*) A system of 3 equations in 5 unknowns that has exactly 1 solution.

Here an example is not sufficient. This is impossible so you need to give a proof that works in all cases.

Proof. Impossible. Either there is a pivot in the final column or there is not. If there is a pivot in the final column, by Theorem 2 the system is not consistent. If there is no pivot in the final column, then there are at most 3 pivots since there are only 3 rows. But there are 5 variables, so at least 2 must be free. In this case by Theorem 2 there are infinitely many solutions. So in either case, there is never exactly one solution. □

6. Section **1.3**, Exercise 10. Easy, but important in the next section!

Solution.

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

7. Prove part (vii) of the Algebraic Properties of $\mathbb{R}^n$ Theorem (p. 27). See the solution to Practice Problem 1 of Section 1.3 and the proof of (vi) from lecture (Friday, January 29th) for examples of how such a proof should go.

Proof. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ and let c and d be scalars. Then by definition of scalar multiplication,

$$c(d\mathbf{u}) = c \begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix} = \begin{bmatrix} c(du_1) \\ \vdots \\ c(du_n) \end{bmatrix}$$

Note the use of parentheses in the multiplication at this stage. Now for the key step: Because multiplication of real numbers is associative, we can rewrite this as

$$= \begin{bmatrix} (cd)u_1 \\ \vdots \\ (cd)u_n \end{bmatrix} = (cd) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = (cd)\mathbf{u}.$$

□

8. Consider the set $H = \left\{ \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ -6 \\ -2 \end{bmatrix} \right\}$. Is the vector $\begin{bmatrix} -32 \\ 4 \\ -7 \end{bmatrix}$ in Span(H)? If it is, write it as a specific linear combination of the vectors in H.

Solution. Form the augmented matrix of the corresponding linear system and reduce:

$$\begin{bmatrix} 4 & -8 & 8 & -32 \\ -4 & 7 & -6 & 4 \\ 2 & -1 & -2 & -7 \end{bmatrix} \sim \begin{bmatrix} 4 & -8 & 8 & -32 \\ 0 & -1 & 2 & -28 \\ 0 & 3 & -6 & 9 \end{bmatrix} \sim \begin{bmatrix} 4 & -8 & 8 & -32 \\ 0 & -1 & 2 & -28 \\ 0 & 0 & 0 & -75 \end{bmatrix}$$

There is a pivot in the rightmost column, so the system has no solution. So the vector is not in Span(H).

9. Consider the set $H = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -5 \end{bmatrix} \right\}$. Is the vector $\begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ in Span(H)? If it is, write it as a specific linear combination of the vectors in H.

Solution. Form the augmented matrix of the corresponding linear system and reduce:

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & -5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

The system is consistent so the vector is in Span $\{H\}$. The solution $x_1 = 5$, $x_2 = 3$, and $x_3 = -2$ provide the **weights** in the combination:

$$5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} -6 \\ 7 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 - 6 + 12 \\ 0 + 9 - 14 \\ 5 - 6 + 10 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix} . \checkmark$$